

Dual Representation of Quasiconvex Conditional Maps

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Abstract

We provide a dual representation of quasiconvex maps $\pi : L_{\mathcal{F}} \rightarrow L_{\mathcal{G}}$, between two lattices of random variables, in terms of conditional expectations. This generalizes the dual representation of quasiconvex real valued functions $\pi : L_{\mathcal{F}} \rightarrow \mathbb{R}$ and the dual representation of conditional convex maps $\pi : L_{\mathcal{F}} \rightarrow L_{\mathcal{G}}$.

Keywords: quasiconvex functions, dual representation, quasiconvex optimization, dynamic risk measures, conditional certainty equivalent.

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1 Introduction¹

Quasiconvex analysis has important applications in several optimization problems in science, economics and in finance, where convexity may be lost due to absence of global risk aversion, as for example in Prospect Theory [KT92].

The first relevant mathematical findings on quasiconvex functions were provided by De Finetti [DF49] and since then many authors, as [Fe49], [Cr77], [Cr80], [ML81], [PP84] and [PV90] - to mention just a few, contributed significantly to the subject. More recently, a Decision Theory complete duality involving quasiconvex real valued functions has been proposed by [CM09]. For a review of quasiconvex analysis and its application and for an exhaustive list of references on this topic we refer to Penot [Pe07].

A function $f : L \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ defined on a vector space L is quasiconvex if for all $c \in \mathbb{R}$ the lower level sets $\{X \in L \mid f(X) \leq c\}$ are convex. In a general setting, the dual representation of such functions was shown by Penot and Volle [PV90]. The following theorem, reformulated in order to be compared to our results, was proved by Volle [Vo98], Th. 3.4. As shown in the

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Appendix 5.2, its proof relies on a straightforward application of Hahn Banach Theorem.

Theorem 1 ([Vo98]) *Let L be a locally convex topological vector space, L' be its dual space and $f : L \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ be quasiconvex and lower semicontinuous. Then*

$$f(X) = \sup_{X' \in L'} R(X'(X), X') \quad (1)$$

where $R : \mathbb{R} \times L' \rightarrow \overline{\mathbb{R}}$ is defined by

$$R(t, X') := \inf_{\xi \in L} \{f(\xi) \mid X'(\xi) \geq t\}.$$

The generality of this theorem rests on the very weak assumptions made on the domain of the function f , i.e. on the space L . On the other hand, the fact that only *real valued* maps are admitted considerably limits its potential applications, specially in a dynamic framework.

To the best of our knowledge, a *conditional* version of this representation is lacking in the literature. When $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ is a filtered probability space, many problems having dynamic features leads to the analysis of maps $\pi : L_t \rightarrow L_s$ between the subspaces $L_t \subseteq L^1(\Omega, \mathcal{F}_t, \mathbb{P})$ and $L_s \subseteq L^0(\Omega, \mathcal{F}_s, \mathbb{P})$, $0 \leq s < t$.

In this paper we consider quasiconvex maps of this form and analyze their dual representation. We provide (see Theorem 8 for the exact statement) a conditional version of (1):

$$\pi(X) = \text{ess} \sup_{Q \in L_t^* \cap \mathcal{P}} R(E_Q[X|\mathcal{F}_s], Q), \quad (2)$$

where

$$R(Y, Q) := \text{ess} \inf_{\xi \in L_t} \{ \pi(\xi) \mid E_Q[\xi|\mathcal{F}_s] \geq_Q Y \}, \quad Y \in L_s,$$

L_t^* is the order continuous dual space of L_t and $\mathcal{P} =: \left\{ \frac{dQ}{d\mathbb{P}} \mid Q \ll \mathbb{P} \right\}$.

Furthermore, we show that if the map π is quasiconvex, monotone and cash additive then it is very easy to derive from (2) the well known representation of a conditional convex risk measure [DS05].

The formula (2) is obtained under quite weak assumptions on the space L_t which allow us to consider maps π defined on the typical spaces used in the literature in this framework: $L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$, $L^p(\Omega, \mathcal{F}_t, \mathbb{P})$, the Orlicz spaces $L^\Psi(\Omega, \mathcal{F}_t, \mathbb{P})$.

We state our results under the assumption that π is lower semicontinuous with respect to the weak topology $\sigma(L_t, L_t^*)$. As shown in Proposition 13 this condition is equivalent to continuity from below, which is the natural requirement in this context.

The proof of our main Theorem 8 is not based on techniques similar to those applied in the quasiconvex real valued case [Vo98], nor to those used for convex conditional maps [DS05]. The idea of the proof is to apply (1) to the real valued quasiconvex map $\pi_A : L_t \rightarrow \overline{\mathbb{R}}$ defined by $\pi_A(X) := \text{ess sup}_{\omega \in A} \pi(X)(\omega)$, $A \in \mathcal{F}_s$, and to approximate $\pi(X)$ with

$$\pi^\Gamma(X) := \sum_{A \in \Gamma} \pi_A(X) \mathbf{1}_A,$$

where Γ is a finite partition of Ω of \mathcal{F}_s measurable sets $A \in \Gamma$. As explained in Section 4.1, some delicate issues arise when one tries to apply this simple and natural idea to prove that:

$$\begin{aligned} & \text{ess sup}_{Q \in L_t^+ \cap \mathcal{P}} \text{ess inf}_{\xi \in L_t} \{ \pi(\xi) | E_Q[\xi | \mathcal{F}_s] \geq_Q E_Q[X | \mathcal{F}_s] \} \\ &= \text{ess inf}_{\Gamma} \text{ess sup}_{Q \in L_t^+ \cap \mathcal{P}} \text{ess inf}_{\xi \in L_t} \{ \pi^\Gamma(\xi) | E_Q[\xi | \mathcal{F}_s] \geq_Q E_Q[X | \mathcal{F}_s] \} \end{aligned} \quad (3)$$

The uniform approximation result here needed is stated in the key Lemma 27 and the Appendix 5.1 is devoted to prove it.

In this paper we limit ourselves to consider conditional maps $\pi : L_t \rightarrow L_s$ and we defer to a forthcoming paper the study of the temporal consistency of the family of maps $(\pi_s)_{s \in [0, t]}$, $\pi_s : L_t \rightarrow L_s$.

As a further motivation for our findings, we give two examples of quasiconvex conditional maps arising in economics and finance, which will also be analyzed in details in a forthcoming paper.

1. *Certainty Equivalent in dynamic settings* . Consider a stochastic dynamic utility (SDU)

$$u : \mathbb{R} \times [0, \infty) \times \Omega \rightarrow \mathbb{R}$$

that satisfies the following conditions: the function $x \rightarrow u(x, t, \omega)$ is strictly increasing and concave on \mathbb{R} , for almost any $\omega \in \Omega$ and for $t \in [0, \infty)$, and $u(x, t, \cdot) \in L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$ for all $(x, t) \in \mathbb{R} \times [0, \infty)$. This functions have been recently considered in [MZ06] and [MSZ08] to develop the theory of forward utility.

In [FM09] we study the *Conditional Certainty Equivalent* (CCE) of a random variable $X \in L_t$, which is defined as the random variable $\pi(X) \in L_s$ solution of the equation:

$$u(\pi(X), s) = E_{\mathbb{P}}[u(X, t) | \mathcal{F}_s].$$

Thus the CCE defines the *valuation* operator

$$\pi : L_t \rightarrow L_s, \quad \pi(X) = u^{-1}(E_{\mathbb{P}}[u(X, t) | \mathcal{F}_s], s).$$

We showed in [FM09] that the CCE, as a map $\pi : L^\infty(\Omega, \mathcal{F}_t, \mathbb{P}) \rightarrow L^\infty(\Omega, \mathcal{F}_s, \mathbb{P})$, is monotone, quasi concave, regular and that for every $X \in L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$

$$\pi(X) = \inf_{Q \in \mathcal{P}} \sup_{\xi \in L^\infty(\mathcal{F}_t)} \{ \pi(\xi) \mid E_Q[\xi | \mathcal{F}_s] =_Q E_Q[X | \mathcal{F}_s] \}. \quad (4)$$

2. Risk measures.

- (a) *Real valued quasiconvex risk measures.* Our interest in quasiconvex analysis was triggered by the recent paper [CV09] on quasiconvex risk measures, where the authors shows that it is reasonable to weaken the convexity axiom in the theory of convex risk measures, introduced in [FS02] and [FR02]. This allows to maintain a good control of the risk, if one also replaces cash additivity by cash subadditivity [ER09].
- (b) *Dynamic risk measures.* As already mentioned the dual representation of a conditional *convex* risk measure can be found in [DS05] and [FP06]. The findings of the present paper can be adapted to prove the dual representation of conditional *quasiconvex* risk measures.

The paper is organized as follows. In Section 2 we introduce the key definitions in order to have all the ingredients to state, in Section 2.1, our main results. Section 3 is a collection of *a priori* properties about the maps we use to obtain the dual representation. Theorem 8, is proved in Section 4 and a brief outline of the proof is there reported to facilitate its understanding. The technical important Lemmas are left to the Appendix, where we also report the proof of Theorem 1.

2 The dual representation

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is fixed throughout the paper and $\mathcal{G} \subseteq \mathcal{F}$ is any sigma algebra contained in \mathcal{F} . As usual we denote with $L^0(\Omega, \mathcal{F}, \mathbb{P})$ the space of \mathcal{F} measurable random variables that are \mathbb{P} a.s. finite.

The $L^p(\Omega, \mathcal{F}, \mathbb{P})$ spaces, $p \in [0, \infty]$, will simply be denoted by L^p , unless it is necessary to specify the sigma algebra, in which case we write $L^p_{\mathcal{F}}$. In presence of an arbitrary measure μ , if confusion may arise, we will explicitly write $=_\mu$ (resp. \geq_μ), meaning μ almost everywhere. Otherwise, all equalities/inequalities among random variables are meant to hold \mathbb{P} -a.s. Moreover the essential (\mathbb{P} almost surely) *supremum* $\text{ess sup}_\lambda(X_\lambda)$ of an arbitrary family of random variables $X_\lambda \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ will be simply denoted by $\sup_\lambda(X_\lambda)$, and similarly for the essential *infimum* (see [FS04] Section A.5 for reference). Here we only notice that $1_A \sup_\lambda(X_\lambda) = \sup_\lambda(1_A X_\lambda)$ for any \mathcal{F} measurable set A . Hereafter the symbol \hookrightarrow denotes inclusion and lattice embedding between two lattices; \vee (resp. \wedge) denotes the essential (\mathbb{P} almost surely) *maximum* (resp. the essential *minimum*) between two random variables, which are the usual lattice operations.

We consider a lattice $L_{\mathcal{F}} := L(\Omega, \mathcal{F}, \mathbb{P}) \subseteq L^0(\Omega, \mathcal{F}, \mathbb{P})$ and a lattice $L_{\mathcal{G}} := L(\Omega, \mathcal{G}, \mathbb{P}) \subseteq L^0(\Omega, \mathcal{G}, \mathbb{P})$ of \mathcal{F} (resp. \mathcal{G}) measurable random variables.

Definition 2 A map $\pi : L_{\mathcal{F}} \rightarrow L_{\mathcal{G}}$ is said to be

(MON) *monotone increasing* if for every $X, Y \in L_{\mathcal{F}}$

$$X \leq Y \quad \Rightarrow \quad \pi(X) \leq \pi(Y) ;$$

(QCO) *quasiconvex* if for every $X, Y \in L_{\mathcal{F}}$, $\Lambda \in L_{\mathcal{G}}^0$ and $0 \leq \Lambda \leq 1$

$$\pi(\Lambda X + (1 - \Lambda)Y) \leq \pi(X) \vee \pi(Y) ;$$

(LSC) τ -*lower semicontinuous* if the set $\{X \in L_{\mathcal{F}} \mid \pi(X) \leq Y\}$ is closed for every $Y \in L_{\mathcal{G}}$ with respect to a topology τ on $L_{\mathcal{F}}$.

Remark 3 As it happens for real valued maps, it is easy to check that the definition of (QCO) is equivalent to the fact that all the lower level sets

$$\mathcal{A}(Y) = \{X \in L_{\mathcal{F}} \mid \pi(X) \leq Y\} \quad \forall Y \in L_{\mathcal{G}}$$

are conditionally convex i.e. for all $X_1, X_2 \in \mathcal{A}(Y)$ and for all \mathcal{G} -measurable r.v. Λ , $0 \leq \Lambda \leq 1$ one has that $\Lambda X_1 + (1 - \Lambda)X_2 \in \mathcal{A}(Y)$.

Definition 4 A vector space $L_{\mathcal{F}} \subseteq L_{\mathcal{F}}^0$ satisfies the property $1_{\mathcal{F}}$ if

$$X \in L_{\mathcal{F}} \text{ and } A \in \mathcal{F} \implies (X \mathbf{1}_A) \in L_{\mathcal{F}}. \quad (1_{\mathcal{F}})$$

Suppose that $L_{\mathcal{F}}$ (resp. $L_{\mathcal{G}}$) satisfies the property $(1_{\mathcal{F}})$ (resp $1_{\mathcal{G}}$). A map $\pi : L_{\mathcal{F}} \rightarrow L_{\mathcal{G}}$ is said to be

(REG) *regular* if for every $X, Y \in L_{\mathcal{F}}$ and $A \in \mathcal{G}$

$$\pi(X \mathbf{1}_A + Y \mathbf{1}_{A^c}) = \pi(X) \mathbf{1}_A + \pi(Y) \mathbf{1}_{A^c}.$$

Remark 5 The assumption (REG) is actually weaker than the assumption

$$\pi(X \mathbf{1}_A) = \pi(X) \mathbf{1}_A \quad \forall A \in \mathcal{G}. \quad (5)$$

As shown in [DS05], (5) always implies (REG), and they are equivalent if and only if $\pi(0) = 0$.

2.1 The representation theorem and its consequences

Standing assumptions

In the sequel of the paper it is assumed that:

- (a) $\mathcal{G} \subseteq \mathcal{F}$ and the lattice $L_{\mathcal{F}}$ (resp. $L_{\mathcal{G}}$) satisfies the property $(1_{\mathcal{F}})$ (resp $1_{\mathcal{G}}$).
- (b) The order continuous dual of $(L_{\mathcal{F}}, \geq)$, denoted by $L_{\mathcal{F}}^* = (L_{\mathcal{F}}, \geq)^*$, is a lattice ([AB05], Th. 8.28 Ogasawara) that satisfies $L_{\mathcal{F}}^* \hookrightarrow L_{\mathcal{F}}^1$ and property $(1_{\mathcal{F}})$.

- (c) The space $L_{\mathcal{F}}$ endowed with the weak topology $\sigma(L_{\mathcal{F}}, L_{\mathcal{F}}^*)$ is a locally convex Riesz space.

The condition (c) requires that the order continuous dual $L_{\mathcal{F}}^*$ is rich enough to separate the points of $L_{\mathcal{F}}$, so that $(L_{\mathcal{F}}, \sigma(L_{\mathcal{F}}, L_{\mathcal{F}}^*))$ becomes a locally convex TVS and Proposition 33 can be applied.

Remark 6 Many important classes of spaces satisfy these conditions, as for example

- The L^p -spaces, $p \in [1, \infty]$: $L_{\mathcal{F}} = L_{\mathcal{F}}^p$, $L_{\mathcal{F}}^* = L_{\mathcal{F}}^q \hookrightarrow L_{\mathcal{F}}^1$.
- The Orlicz spaces L^{Ψ} for any Young function Ψ : $L_{\mathcal{F}} = L_{\mathcal{F}}^{\Psi}$, $L_{\mathcal{F}}^* = L_{\mathcal{F}}^{\Psi^*} \hookrightarrow L_{\mathcal{F}}^1$, where Ψ^* denotes the conjugate function of Ψ ;
- The Morse subspace M^{Ψ} of the Orlicz space L^{Ψ} , for any continuous Young function Ψ : $L_{\mathcal{F}} = M_{\mathcal{F}}^{\Psi}$, $L_{\mathcal{F}}^* = L_{\mathcal{F}}^{\Psi^*} \hookrightarrow L_{\mathcal{F}}^1$.

Set

$$\mathcal{P} =: \left\{ \frac{dQ}{d\mathbb{P}} \mid Q \ll \mathbb{P} \text{ and } Q \text{ probability} \right\} = \{ \xi' \in L_+^1 \mid E_{\mathbb{P}}[\xi'] = 1 \}$$

From now on we will write with a slight abuse of notation $Q \in L_{\mathcal{F}}^* \cap \mathcal{P}$ instead of $\frac{dQ}{d\mathbb{P}} \in L_{\mathcal{F}}^* \cap \mathcal{P}$. Define for $X \in L_{\mathcal{F}}$ and $Q \in L_{\mathcal{F}}^* \cap \mathcal{P}$

$$K(X, Q) := \inf_{\xi \in L_{\mathcal{F}}} \{ \pi(\xi) \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}] \}$$

and notice that $K(X, Q)$ depends on X only through $E_Q[X|\mathcal{G}]$.

Remark 7 Since the order continuous functional on $L_{\mathcal{F}}$ are contained in L^1 , then $Q(\xi) := E_Q[\xi]$ is well defined and finite for every $\xi \in L_{\mathcal{F}}$ and $Q \in L_{\mathcal{F}}^* \cap \mathcal{P}$. In particular this and $(1_{\mathcal{F}})$ implies that $E_Q[\xi|\mathcal{G}]$ is well defined. Moreover, since $L_{\mathcal{F}}^* \hookrightarrow L_{\mathcal{F}}^1$ satisfies property $(1_{\mathcal{F}})$ then $\frac{dQ}{d\mathbb{P}} 1_A \in L_{\mathcal{F}}^*$ whenever $Q \in L_{\mathcal{F}}^*$ and $A \in \mathcal{F}$.

Theorem 8 If $\pi : L_{\mathcal{F}} \rightarrow L_{\mathcal{G}}$ is (MON), (QCO), (REG) and $\sigma(L_{\mathcal{F}}, L_{\mathcal{F}}^*)$ -LSC then

$$\pi(X) = \sup_{Q \in L_{\mathcal{F}}^* \cap \mathcal{P}} K(X, Q). \quad (6)$$

Notice that in (6) the *supremum* is taken over the set $L_{\mathcal{F}}^* \cap \mathcal{P}$. In the following corollary, proved in Section 4.2, we show that we can match the conditional convex dual representation, restricting our optimization problem over the set

$$\mathcal{P}_{\mathcal{G}} =: \left\{ \frac{dQ}{d\mathbb{P}} \mid Q \in \mathcal{P} \text{ and } Q = \mathbb{P} \text{ on } \mathcal{G} \right\}.$$

Clearly, when $Q \in \mathcal{P}_{\mathcal{G}}$ then $L^0(\Omega, \mathcal{G}, \mathbb{P}) = L^0(\Omega, \mathcal{G}, Q)$ and comparison of \mathcal{G} measurable random variables is understood to hold indifferently for \mathbb{P} or Q .

Corollary 9 *Under the same hypothesis of Theorem 8, suppose that for $X \in L_{\mathcal{F}}$ there exists $\eta \in L_{\mathcal{F}}$ and $\delta > 0$ such that $\mathbb{P}(\pi(\eta) + \delta < \pi(X)) = 1$. Then*

$$\pi(X) = \sup_{Q \in L_{\mathcal{F}}^* \cap \mathcal{P}_{\mathcal{G}}} K(X, Q).$$

Remark 10 *It's worth to be observed that actually the assumption (MON) is only used for obtaining the dual representation (6) over the set of positive elements of the dual space, i.e. on probability measures (see Proposition 33). On the other hand, for $\xi' \in L_{\mathcal{F}}^* \cap L_{\mathcal{F}}^1$, we could define a generalized conditional expected value $E_{\mu}[X|\mathcal{G}] =_{\mu} E_{\mathbb{P}}[\xi' X|\mathcal{G}] \cdot E_{\mathbb{P}}[\xi'|\mathcal{G}]^{-1}$, where μ is a finite signed measure whose density is $\frac{d\mu}{d\mathbb{P}} = \xi'$ and drop the (MON) assumption in Theorem 8. Only in the next three results the (MON) plays a role.*

Definition 11 *We say that $\pi : L_{\mathcal{F}} \rightarrow L_{\mathcal{G}}$ is*

(CFB) *continuous from below if*

$$X_n \uparrow X \quad \mathbb{P} \text{ a.s.} \quad \Rightarrow \quad \pi(X_n) \uparrow \pi(X) \quad \mathbb{P} \text{ a.s.}$$

In [BF09] it is proved the equivalence between: (CFB), order lsc and $\sigma(L_{\mathcal{F}}, L_{\mathcal{F}}^*)$ -(LSC), for monotone convex real valued functions. In the next proposition we show that this equivalence remains true for monotone quasiconvex conditional maps, under the same assumption on the topology $\sigma(L_{\mathcal{F}}, L_{\mathcal{F}}^*)$ adopted in [BF09].

Definition 12 ([BF09]) *A linear topology τ on a Riesz space has the C-property if $X_{\alpha} \xrightarrow{\tau} X$ implies the existence of a sequence $\{X_{\alpha_n}\}_n$ and a convex combination $Z_n \in \text{conv}(X_{\alpha_n}, \dots)$ such that $Z_n \xrightarrow{\circ} X$.*

As explained in [BF09], the assumption that $\sigma(L_{\mathcal{F}}, L_{\mathcal{F}}^*)$ has the C-property is very weak and is satisfied in all cases of interest. When this is the case, in Theorem 8 the $\sigma(L_{\mathcal{F}}, L_{\mathcal{F}}^*)$ -(LSC) condition can be replaced by (CFB), which is often easy to check.

Proposition 13 *Suppose that $\sigma(L_{\mathcal{F}}, L_{\mathcal{F}}^*)$ satisfies the C-property and that $L_{\mathcal{F}}$ is order complete. Given $\pi : L_{\mathcal{F}} \rightarrow L_{\mathcal{G}}$ satisfying (MON) and (QCO) we have: (i) π is $\sigma(L_{\mathcal{F}}, L_{\mathcal{F}}^*)$ -(LSC) if and only if (ii) π is (CFB).*

Proof. Recall that a sequence $\{X_n\} \subseteq L_{\mathcal{F}}$ order converge to $X \in L_{\mathcal{F}}$, $X_n \xrightarrow{\circ} X$, if there exists a sequence $\{Y_n\} \subseteq L_{\mathcal{F}}$ satisfying $Y_n \downarrow 0$ and $|X - X_n| \leq Y_n$.

(i) \Rightarrow (ii): Consider $X_n \uparrow X$. Since $X_n \uparrow X$ implies $X_n \xrightarrow{\circ} X$, then for every order continuous $Z \in L_{\mathcal{F}}^*$ the convergence $Z(X_n) \rightarrow Z(X)$ holds. From $L_{\mathcal{F}}^* \hookrightarrow L_{\mathcal{F}}^1$

$$E_{\mathbb{P}}[ZX_n] \rightarrow E_{\mathbb{P}}[ZX] \quad \forall Z \in L_{\mathcal{F}}^*$$

and we deduce that $X_n \xrightarrow{\sigma(L_{\mathcal{F}}, L_{\mathcal{F}}^*)} X$.

(MON) implies $\pi(X_n) \uparrow$ and $p := \lim_n \pi(X_n) \leq \pi(X)$. The lower level set $\mathcal{A}_p = \{\xi \in L_{\mathcal{F}} \mid \pi(\xi) \leq p\}$ is $\sigma(L_{\mathcal{F}}, L_{\mathcal{F}}^*)$ closed and then $X \in \mathcal{A}_p$, i.e. $\pi(X) = p$.

(ii) \Rightarrow (i): First we prove that if $X_n \xrightarrow{o} X$ then $\pi(X) \leq \liminf_n \pi(X_n)$. Define $Z_n := (\inf_{k \geq n} X_k) \wedge X$ and note that $X - Y_n \leq X_n \leq X + Y_n$ implies

$$X \geq Z_n = \left(\inf_{k \geq n} X_k \right) \wedge X \geq \left(\inf_{k \geq n} (-Y_k) + X \right) \wedge X \uparrow X$$

i.e. $Z_n \uparrow X$. We actually have from (MON) $Z_n \leq X_n$ implies $\pi(Z_n) \leq \pi(X_n)$ and from (CFB) $\pi(X) = \lim_n \pi(Z_n) \leq \liminf_n \pi(X_n)$ which was our first claim. For $Y \in L_{\mathcal{G}}$ consider $\mathcal{A}_Y = \{\xi \in L_{\mathcal{F}} \mid \pi(\xi) \leq Y\}$ and a net $\{X_\alpha\} \subseteq L_{\mathcal{F}}$ such that $X_\alpha \xrightarrow{\sigma(L_{\mathcal{F}}, L_{\mathcal{F}}^*)} X \in L_{\mathcal{F}}$. Since $L_{\mathcal{F}}$ satisfies the C-property, there exists $Y_n \in \text{Conv}(X_{\alpha_n}, \dots)$ such $Y_n \xrightarrow{o} X$. The property (QCO) implies that \mathcal{A}_Y is convex and then $\{Y_n\} \subseteq \mathcal{A}_Y$. Applying the first step we get

$$\pi(X) \leq \liminf_n \pi(Y_n) \leq Y \quad \text{i.e. } X \in \mathcal{A}_Y$$

■

In the following Lemma and Corollary, proved in Section 3.2, we show that the (MON) property implies that the constraint $E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}]$ may be restricted to $E_Q[\xi|\mathcal{G}] =_Q E_Q[X|\mathcal{G}]$ and that we may recover the dual representation of a dynamic risk measure. When $Q \in L_{\mathcal{F}}^* \cap \mathcal{P}_{\mathcal{G}}$ the previous inequality/equality may be equivalently intended Q -a.s. or \mathbb{P} -a.s. and so we do not need any more to emphasize this in the notations.

Lemma 14 *Suppose that for every $Q \in L_{\mathcal{F}}^* \cap \mathcal{P}_{\mathcal{G}}$ and $\xi \in L_{\mathcal{F}}$ we have $E_Q[\xi|\mathcal{G}] \in L_{\mathcal{F}}$. If $Q \in L_{\mathcal{F}}^* \cap \mathcal{P}_{\mathcal{G}}$ and if $\pi : L_{\mathcal{F}} \rightarrow L_{\mathcal{G}}$ is (MON) and (REG) then*

$$K(X, Q) = \inf_{\xi \in L_{\mathcal{F}}} \{\pi(\xi) \mid E_Q[\xi|\mathcal{G}] = E_Q[X|\mathcal{G}]\}. \quad (7)$$

Definition 15 *Suppose that $\pi : L_{\mathcal{F}} \rightarrow L_{\mathcal{G}}$ is convex. The conditional Fenchel convex conjugate π^* of π is given, for $Q \in L_{\mathcal{F}}^* \cap \mathcal{P}_{\mathcal{G}}$, by the extended valued \mathcal{G} -measurable random variable:*

$$\pi^*(Q) = \sup_{\xi \in L_{\mathcal{F}}} \{E_Q[\xi|\mathcal{G}] - \pi(\xi)\}.$$

A map $\pi : L_{\mathcal{F}} \rightarrow L_{\mathcal{G}}$ is said to be

(CAS) *cash invariant if for all $X \in L_{\mathcal{F}}$ and $\Lambda \in L_{\mathcal{G}}$*

$$\pi(X + \Lambda) = \pi(X) + \Lambda.$$

In the literature [FR04], [DS05], [FP06] a map $\rho : L_{\mathcal{F}} \rightarrow L_{\mathcal{G}}$ that is monotone (decreasing), convex, cash invariant and regular is called a *convex conditional (or dynamic) risk measure*. As a corollary of our main theorem, we deduce immediately the dual representation of a map π satisfying (CAS), in terms of the Fenchel conjugate π^* , in agreement with [DS05]. Of course, this is of no surprise since the (CAS) and (QCO) properties imply convexity, but it supports the correctness of our dual representation.

Corollary 16 Suppose that for every $Q \in L_{\mathcal{F}}^* \cap \mathcal{P}_{\mathcal{G}}$ and $\xi \in L_{\mathcal{F}}$ we have $E_Q[\xi|\mathcal{G}] \in L_{\mathcal{F}}$.

(i) If $Q \in L_{\mathcal{F}}^* \cap \mathcal{P}_{\mathcal{G}}$ and if $\pi : L_{\mathcal{F}} \rightarrow L_{\mathcal{G}}$ is (MON), (REG) and (CAS) then

$$K(X, Q) = E_Q[X|\mathcal{G}] - \pi^*(Q). \quad (8)$$

(ii) Under the same assumptions of Theorem 8 and if π satisfies in addition (CAS) then

$$\pi(X) = \sup_{Q \in L_{\mathcal{F}}^* \cap \mathcal{P}_{\mathcal{G}}} \{E_Q[X|\mathcal{G}] - \pi^*(Q)\}.$$

3 Preliminary results

In the sequel of the paper it is always assumed that $\pi : L_{\mathcal{F}} \rightarrow L_{\mathcal{G}}$ satisfies (REG).

3.1 Properties of $R(Y, \xi')$

To any $\xi' \in L_{\mathcal{F}}^* \cap (L_{\mathcal{F}}^1)_+$ we may associate a measure μ such that $\frac{d\mu}{d\mathbb{P}} = \xi'$. Given an arbitrary $Y \in L_{\mathcal{G}}^0$, define:

$$\mathcal{A}(Y, \xi') := \{\pi(\xi) \mid \xi \in L_{\mathcal{F}} \text{ } E_{\mathbb{P}}[\xi'\xi|\mathcal{G}] \geq_{\mu} Y\},$$

$$R(Y, \xi') := \inf_{\xi \in L_{\mathcal{F}}} \{\pi(\xi) \mid E_{\mathbb{P}}[\xi'\xi|\mathcal{G}] \geq_{\mu} Y\} = \inf \mathcal{A}(Y, \xi').$$

Lemma 17 For every $Y \in L_{\mathcal{G}}^0$ and $\xi' \in L_{\mathcal{F}}^* \cap (L_{\mathcal{F}}^1)_+$ the set $\mathcal{A}(Y, \xi')$ is downward directed and therefore there exists a sequence $\{\eta_m\}_{m=1}^{\infty} \in L_{\mathcal{F}}$ such that $E_{\mathbb{P}}[\xi'\eta_m|\mathcal{G}] \geq_{\mu} Y$ and as $m \uparrow \infty$, $\pi(\eta_m) \downarrow R(Y, \xi')$.

Proof. We have to prove that for every $\pi(\xi_1), \pi(\xi_2) \in \mathcal{A}(Y, \xi')$ there exists $\pi(\xi^*) \in \mathcal{A}(Y, \xi')$ such that $\pi(\xi^*) \leq \min\{\pi(\xi_1), \pi(\xi_2)\}$. Consider the \mathcal{G} -measurable set $G = \{\pi(\xi_1) \leq \pi(\xi_2)\}$ then

$$\min\{\pi(\xi_1), \pi(\xi_2)\} = \pi(\xi_1)\mathbf{1}_G + \pi(\xi_2)\mathbf{1}_{G^c} = \pi(\xi_1\mathbf{1}_G + \xi_2\mathbf{1}_{G^c}) = \pi(\xi^*),$$

where $\xi^* = \xi_1\mathbf{1}_G + \xi_2\mathbf{1}_{G^c}$. Since $E_{\mathbb{P}}[\xi'\xi^*|\mathcal{G}] = E_{\mathbb{P}}[\xi'\xi_1|\mathcal{G}]\mathbf{1}_G + E_{\mathbb{P}}[\xi'\xi_2|\mathcal{G}]\mathbf{1}_{G^c}$ and $\mu \ll \mathbb{P}$ together imply $E_{\mathbb{P}}[\xi'\xi^*|\mathcal{G}] =_{\mu} E_{\mathbb{P}}[\xi'\xi_1|\mathcal{G}]\mathbf{1}_G + E_{\mathbb{P}}[\xi'\xi_2|\mathcal{G}]\mathbf{1}_{G^c} \geq_{\mu} Y$, we can deduce $\pi(\xi^*) \in \mathcal{A}(Y, \xi')$. ■

Lemma 18 Properties of $R(Y, \xi')$. Let $\xi' \in L_{\mathcal{F}}^* \cap (L_{\mathcal{F}}^1)_+$.

- i) $R(\cdot, \xi')$ is monotone
- ii) $R(\lambda Y, \lambda \xi') = R(Y, \xi')$ for any $\lambda > 0$, $Y \in L_{\mathcal{G}}$.
- iii) For every $A \in \mathcal{G}$, $X \in L_{\mathcal{F}}$ and $Y =_{\mu} E_{\mathbb{P}}[X\xi'|\mathcal{G}]$

$$R(Y, \xi')\mathbf{1}_A = \inf_{\xi \in L_{\mathcal{F}}} \{\pi(\xi)\mathbf{1}_A \mid E_{\mathbb{P}}[\xi'\xi|\mathcal{G}] \geq_{\mu} Y\} \quad (9)$$

$$= \inf_{\xi \in L_{\mathcal{F}}} \{\pi(\xi)\mathbf{1}_A \mid E_{\mathbb{P}}[\xi'\xi\mathbf{1}_A|\mathcal{G}] \geq_{\mu} Y\mathbf{1}_A\}, \quad (10)$$

- iv) For every $Y_1, Y_2 \in L_{\mathcal{G}}$
- (a) $R(Y_1, \xi') \wedge R(Y_2, \xi') = R(Y_1 \wedge Y_2, \xi')$
 - (b) $R(Y_1, \xi') \vee R(Y_2, \xi') = R(Y_1 \vee Y_2, \xi')$
- v) The map $R(Y, \xi')$ is quasi-affine with respect to Y in the sense that for every $Y_1, Y_2, \Lambda \in L_{\mathcal{G}}$ and $0 \leq \Lambda \leq 1$, we have
- $R(\Lambda Y_1 + (1 - \Lambda)Y_2, \xi') \geq R(Y_1, \xi') \wedge R(Y_2, \xi') \quad (\text{quasiconcavity})$
 - $R(\Lambda Y_1 + (1 - \Lambda)Y_2, \xi') \leq R(Y_1, \xi') \vee R(Y_2, \xi') \quad (\text{quasiconvexity}).$

Proof. Since $\pi(\xi \mathbf{1}_A) - \pi(0) \mathbf{1}_{A^c} = \pi(\xi) \mathbf{1}_A$, w.l.o.g. we may assume in the sequel of this proof that $\pi(0) = 0$ and so $\pi(\xi \mathbf{1}_A) = \pi(\xi) \mathbf{1}_A$.

(i) and (ii) are trivial.

(iii) By definition of the essential infimum one easily deduce (9). Let $\frac{d\mu}{d\mathbb{P}} = \xi'$. To prove (10), for every $\xi \in L_{\mathcal{F}}$ such that $E_{\mathbb{P}}[\xi' \xi \mathbf{1}_A | \mathcal{G}] \geq_{\mu} Y \mathbf{1}_A$ we define the random variable $\eta = \xi \mathbf{1}_A + X \mathbf{1}_{A^c}$ which satisfies $E_{\mathbb{P}}[\xi' \eta | \mathcal{G}] \geq_{\mu} Y$. In fact since $\mu \ll \mathbb{P}$ we have that $E_{\mathbb{P}}[\xi' \eta | \mathcal{G}] = E_{\mathbb{P}}[\xi' \xi | \mathcal{G}] \mathbf{1}_A + E_{\mathbb{P}}[\xi' X | \mathcal{G}] \mathbf{1}_{A^c}$ implies

$$E_{\mathbb{P}}[\xi' \eta | \mathcal{G}] =_{\mu} E_{\mathbb{P}}[\xi' \xi | \mathcal{G}] \mathbf{1}_A + E_{\mathbb{P}}[\xi' X | \mathcal{G}] \mathbf{1}_{A^c} \geq_{\mu} Y.$$

Therefore

$$\{\eta \mathbf{1}_A \mid \eta \in L_{\mathcal{F}}, E_{\mathbb{P}}[\xi' \eta | \mathcal{G}] \geq_{\mu} Y\} = \{\xi \mathbf{1}_A \mid \xi \in L_{\mathcal{F}}, E_{\mathbb{P}}[\xi' \xi \mathbf{1}_A | \mathcal{G}] \geq_{\mu} Y \mathbf{1}_A\}$$

Hence from (9):

$$\begin{aligned} \mathbf{1}_A R(Y, \xi') &= \inf_{\eta \in L_{\mathcal{F}}} \{\pi(\eta \mathbf{1}_A) \mid E_{\mathbb{P}}[\xi' \eta | \mathcal{G}] \geq_{\mu} Y\} \\ &= \inf_{\xi \in L_{\mathcal{F}}} \{\pi(\xi \mathbf{1}_A) \mid E_{\mathbb{P}}[\xi' \xi \mathbf{1}_A | \mathcal{G}] \geq_{\mu} Y \mathbf{1}_A\} \end{aligned}$$

and (10) follows.

iv) a): Since $R(\cdot, \xi')$ is monotone, the inequalities $R(Y_1, \xi') \wedge R(Y_2, \xi') \geq R(Y_1 \wedge Y_2, \xi')$ and $R(Y_1, \xi') \vee R(Y_2, \xi') \leq R(Y_1 \vee Y_2, \xi')$ are always true.

To show the opposite inequalities, define the \mathcal{G} -measurable sets: $B := \{R(Y_1, \xi') \leq R(Y_2, \xi')\}$ and $A := \{Y_1 \leq Y_2\}$ so that

$$\begin{aligned} R(Y_1, \xi') \wedge R(Y_2, \xi') &= R(Y_1, \xi') \mathbf{1}_B + R(Y_2, \xi') \mathbf{1}_{B^c} \leq R(Y_1, \xi') \mathbf{1}_A + R(Y_2, \xi') \mathbf{1}_{A^c} \\ R(Y_1, \xi') \vee R(Y_2, \xi') &= R(Y_1, \xi') \mathbf{1}_{B^c} + R(Y_2, \xi') \mathbf{1}_B \geq R(Y_1, \xi') \mathbf{1}_{A^c} + R(Y_2, \xi') \mathbf{1}_A \end{aligned} \tag{11}$$

Set: $D(A, Y) = \{\xi \mathbf{1}_A \mid \xi \in L_{\mathcal{F}}, E_{\mathbb{P}}[\xi' \xi \mathbf{1}_A | \mathcal{G}] \geq_{\mu} Y \mathbf{1}_A\}$ and check that

$$D(A, Y_1) + D(A^c, Y_2) = \{\xi \in L_{\mathcal{F}} \mid E_{\mathbb{P}}[\xi' \xi | \mathcal{G}] \geq_{\mu} Y_1 \mathbf{1}_A + Y_2 \mathbf{1}_{A^c}\} := D$$

From (11) and using (10) we get:

$$\begin{aligned}
R(Y_1, \xi') \wedge R(Y_2, \xi') &\leq R(Y_1, \xi') \mathbf{1}_A + R(Y_2, \xi') \mathbf{1}_{A^C} \\
&= \inf_{\xi \mathbf{1}_A \in D(A, Y_1)} \{\pi(\xi \mathbf{1}_A)\} + \inf_{\eta \mathbf{1}_{A^C} \in D(A^C, Y_2)} \{\pi(\eta \mathbf{1}_{A^C})\} \\
&= \inf_{\substack{\xi \mathbf{1}_A \in D(A, Y_1) \\ \eta \mathbf{1}_{A^C} \in D(A^C, Y_2)}} \{\pi(\xi \mathbf{1}_A) + \pi(\eta \mathbf{1}_{A^C})\} \\
&= \inf_{(\xi \mathbf{1}_A + \eta \mathbf{1}_{A^C}) \in D(A, Y_1) + D(A^C, Y_2)} \{\pi(\xi \mathbf{1}_A + \eta \mathbf{1}_{A^C})\} \\
&= \inf_{\xi \in D} \{\pi(\xi)\} = R(Y_1 \mathbf{1}_A + Y_2 \mathbf{1}_{A^C}, \xi') = R(Y_1 \wedge Y_2, \xi').
\end{aligned}$$

Simile modo: iv) b).

(v) From the monotonicity of $R(\cdot, \xi')$, $R(Y_1 \wedge Y_2, \xi') \leq R(\Lambda Y_1 + (1 - \Lambda) Y_2, \xi')$ (resp. $R(Y_1 \vee Y_2, \xi') \geq R(\Lambda Y_1 + (1 - \Lambda) Y_2, \xi')$) and then the thesis follows from iv). ■

3.2 Properties of $K(X, Q)$

For $\xi' \in L_{\mathcal{F}}^* \cap (L_{\mathcal{F}}^1)_+$ and $X \in L_{\mathcal{F}}$

$$R(E_{\mathbb{P}}[\xi' X | \mathcal{G}], \xi') = \inf_{\xi \in L_{\mathcal{F}}} \{\pi(\xi) \mid E_{\mathbb{P}}[\xi' \xi | \mathcal{G}] \geq_{\mu} E_{\mathbb{P}}[\xi' X | \mathcal{G}]\} = K(X, \xi').$$

Notice that $K(X, \xi') = K(X, \lambda \xi')$ for every $\lambda > 0$ and thus we can consider $K(X, \xi')$, $\xi' \neq 0$, always defined on the normalized elements $Q \in L_{\mathcal{F}}^* \cap \mathcal{P}$.

Moreover, from $E_{\mathbb{P}}\left[\frac{dQ}{d\mathbb{P}} X | \mathcal{G}\right] =_Q E_{\mathbb{P}}\left[\frac{dQ}{d\mathbb{P}} \mid \mathcal{G}\right] E_Q[X | \mathcal{G}]$ and $E_{\mathbb{P}}\left[\frac{dQ}{d\mathbb{P}} \mid \mathcal{G}\right] >_Q 0$ we deduce:

$$E_{\mathbb{P}}\left[\frac{dQ}{d\mathbb{P}} \xi \mid \mathcal{G}\right] \geq_Q E_{\mathbb{P}}\left[\frac{dQ}{d\mathbb{P}} X \mid \mathcal{G}\right] \iff E_Q[\xi | \mathcal{G}] \geq_Q E_Q[X | \mathcal{G}].$$

For $Q \in L_{\mathcal{F}}^* \cap \mathcal{P}$ we then set:

$$K(X, Q) := \inf_{\xi \in L_{\mathcal{F}}} \{\pi(\xi) \mid E_Q[\xi | \mathcal{G}] \geq_Q E_Q[X | \mathcal{G}]\} = R\left(E_{\mathbb{P}}\left[\frac{dQ}{d\mathbb{P}} X \mid \mathcal{G}\right], \frac{dQ}{d\mathbb{P}}\right).$$

Lemma 19 *Properties of $K(X, Q)$. Let $Q \in L_{\mathcal{F}}^* \cap \mathcal{P}$ and $X \in L_{\mathcal{F}}$.*

- i) $K(\cdot, Q)$ is monotone and quasi affine.
- ii) $K(X, \cdot)$ is positively homogeneous.
- iii) $K(X, Q) \mathbf{1}_A = \inf_{\xi \in L_{\mathcal{F}}} \{\pi(\xi) \mathbf{1}_A \mid E_Q[\xi \mathbf{1}_A | \mathcal{G}] \geq_Q E_Q[X \mathbf{1}_A | \mathcal{G}]\}$ for all $A \in \mathcal{G}$.
- iv) There exists a sequence $\left\{\xi_m^Q\right\}_{m=1}^{\infty} \in L_{\mathcal{F}}$ such that

$$E_Q[\xi_m^Q | \mathcal{G}] \geq_Q E_Q[X | \mathcal{G}] \quad \forall m \geq 1, \quad \pi(\xi_m^Q) \downarrow K(X, Q) \quad \text{as } m \uparrow \infty.$$

v) The set $\mathcal{K} = \{K(X, Q) \mid Q \in L_{\mathcal{F}}^* \cap \mathcal{P}\}$ is upward directed, i.e. for every $K(X, Q_1), K(X, Q_2) \in \mathcal{K}$ there exists $K(X, \widehat{Q}) \in \mathcal{K}$ such that $K(X, \widehat{Q}) \geq K(X, Q_1) \vee K(X, Q_2)$.

vi) Let Q_1 and Q_2 be elements of $L_{\mathcal{F}}^* \cap \mathcal{P}$ and $B \in \mathcal{G}$. If $\frac{dQ_1}{d\mathbb{P}} \mathbf{1}_B = \frac{dQ_2}{d\mathbb{P}} \mathbf{1}_B$ then $K(X, Q_1) \mathbf{1}_B = K(X, Q_2) \mathbf{1}_B$.

Proof. The monotonicity property in (i), (ii) and (iii) are trivial; from Lemma 18 v) it follows that $K(\cdot, Q)$ is quasi affine; (iv) is an immediate consequence of Lemma 17.

(v) Define $F = \{K(X, Q_1) \geq K(X, Q_2)\}$ and let \widehat{Q} given by $\frac{d\widehat{Q}}{d\mathbb{P}} := \mathbf{1}_F \frac{dQ_1}{d\mathbb{P}} + \mathbf{1}_{F^c} \frac{dQ_2}{d\mathbb{P}}$; up to a normalization factor (from property (ii)) we may suppose $\widehat{Q} \in L_{\mathcal{F}}^* \cap \mathcal{P}$. We need to show that

$$K(X, \widehat{Q}) = K(X, Q_1) \vee K(X, Q_2) = K(X, Q_1)\mathbf{1}_F + K(X, Q_2)\mathbf{1}_{F^c}.$$

From $E_{\widehat{Q}}[\xi|\mathcal{G}] =_{\widehat{Q}} E_{Q_1}[\xi|\mathcal{G}]\mathbf{1}_F + E_{Q_2}[\xi|\mathcal{G}]\mathbf{1}_{F^c}$ we get $E_{\widehat{Q}}[\xi|\mathcal{G}]\mathbf{1}_F =_{Q_1} E_{Q_1}[\xi|\mathcal{G}]\mathbf{1}_F$ and $E_{\widehat{Q}}[\xi|\mathcal{G}]\mathbf{1}_{F^c} =_{Q_2} E_{Q_2}[\xi|\mathcal{G}]\mathbf{1}_{F^c}$. In the second place, for $i = 1, 2$, consider the sets

$$\widehat{A} = \{\xi \in L_{\mathcal{F}} \mid E_{\widehat{Q}}[\xi|\mathcal{G}] \geq_{\widehat{Q}} E_{\widehat{Q}}[X|\mathcal{G}]\} \quad A_i = \{\xi \in L_{\mathcal{F}} \mid E_{Q_i}[\xi|\mathcal{G}] \geq_{Q_i} E_{Q_i}[X|\mathcal{G}]\}.$$

For every $\xi \in A_1$ define $\eta = \xi\mathbf{1}_F + X\mathbf{1}_{F^c}$

$$\begin{aligned} Q_1 << \mathbb{P} &\Rightarrow \quad \eta\mathbf{1}_F =_{Q_1} \xi\mathbf{1}_F \Rightarrow \quad E_{\widehat{Q}}[\eta|\mathcal{G}]\mathbf{1}_F \geq_{\widehat{Q}} E_{\widehat{Q}}[X|\mathcal{G}]\mathbf{1}_F \\ Q_2 << \mathbb{P} &\Rightarrow \quad \eta\mathbf{1}_{F^c} =_{Q_2} X\mathbf{1}_{F^c} \Rightarrow \quad E_{\widehat{Q}}[\eta|\mathcal{G}]\mathbf{1}_{F^c} =_{\widehat{Q}} E_{\widehat{Q}}[X|\mathcal{G}]\mathbf{1}_{F^c} \end{aligned}$$

Then $\eta \in \widehat{A}$ and $\pi(\xi)\mathbf{1}_F = \pi(\xi\mathbf{1}_F) - \pi(0)\mathbf{1}_{F^c} = \pi(\eta\mathbf{1}_F) - \pi(0)\mathbf{1}_{F^c} = \pi(\eta)\mathbf{1}_F$. Viceversa, for every $\eta \in \widehat{A}$ define $\xi = \eta\mathbf{1}_F + X\mathbf{1}_{F^c}$. Then $\xi \in A_1$ and again $\pi(\xi)\mathbf{1}_F = \pi(\eta)\mathbf{1}_F$. Hence

$$\inf_{\xi \in A_1} \pi(\xi)\mathbf{1}_F = \inf_{\eta \in \widehat{A}} \pi(\eta)\mathbf{1}_F.$$

In a similar way: $\inf_{\xi \in A_2} \pi(\xi)\mathbf{1}_{F^c} = \inf_{\eta \in \widehat{A}} \pi(\eta)\mathbf{1}_{F^c}$ and we can finally deduce $K(X, Q_1) \vee K(X, Q_2) = K(X, \widehat{Q})$.

(vi). By the same argument used in (v), it can be shown that $\inf_{\xi \in A_1} \pi(\xi)\mathbf{1}_B = \inf_{\xi \in A_2} \pi(\xi)\mathbf{1}_B$ and the thesis. ■

Proof of Lemma 14. Let us denote with $r(X, Q)$ the right hand side of equation (7) and notice that $K(X, Q) \leq r(X, Q)$. By contradiction, suppose that $\mathbb{P}(A) > 0$ where $A =: \{K(X, Q) < r(X, Q)\}$. As shown in Lemma 19 iv), there exists a r.v. $\xi \in L_{\mathcal{F}}$ satisfying the following conditions

- $E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}]$ and $Q(E_Q[\xi|\mathcal{G}] > E_Q[X|\mathcal{G}]) > 0$.
- $K(X, Q)(\omega) \leq \pi(\xi)(\omega) < r(X, Q)(\omega)$ for \mathbb{P} -almost every $\omega \in B \subseteq A$ and $\mathbb{P}(B) > 0$.

Set $Z =_Q E_Q[\xi - X|\mathcal{G}]$. By assumption, $Z \in L_{\mathcal{F}}$ and it satisfies $Z \geq_Q 0$ and, since $Q \in \mathcal{P}_{\mathcal{G}}$, $Z \geq 0$. Then, thanks to (MON), $\pi(\xi) \geq \pi(\xi - Z)$. From $E_Q[\xi - Z|\mathcal{G}] =_Q E_Q[X|\mathcal{G}]$ we deduce:

$$K(X, Q)(\omega) \leq \pi(\xi)(\omega) < r(X, Q)(\omega) \leq \pi(\xi - Z)(\omega) \text{ for } \mathbb{P}\text{-a.e. } \omega \in B,$$

which is a contradiction. ■

Proof of Corollary 16. The (CAS) property implies that for every $X \in L_{\mathcal{F}}$ and $\delta > 0$, $\mathbb{P}(\pi(X - 2\delta) + \delta < \pi(X)) = 1$. So the hypothesis of Corollary 9 holds true and we only need to prove (8), since (ii) is a consequence of (i) and Corollary 9. Let $Q \in L_{\mathcal{F}}^* \cap \mathcal{P}_{\mathcal{G}}$. Applying Lemma 14 we deduce:

$$\begin{aligned}
K(X, Q) &= \inf_{\xi \in L_{\mathcal{F}}} \{ \pi(\xi) \mid E_Q[\xi|\mathcal{G}] =_Q E_Q[X|\mathcal{G}] \} \\
&= E_Q[X|\mathcal{G}] + \inf_{\xi \in L_{\mathcal{F}}} \{ \pi(\xi) - E_Q[X|\mathcal{G}] \mid E_Q[\xi|\mathcal{G}] =_Q E_Q[X|\mathcal{G}] \} \\
&= E_Q[X|\mathcal{G}] + \inf_{\xi \in L_{\mathcal{F}}} \{ \pi(\xi) - E_Q[\xi|\mathcal{G}] \mid E_Q[\xi|\mathcal{G}] =_Q E_Q[X|\mathcal{G}] \} \\
&= E_Q[X|\mathcal{G}] - \sup_{\xi \in L_{\mathcal{F}}} \{ E_Q[\xi|\mathcal{G}] - \pi(\xi) \mid E_Q[\xi|\mathcal{G}] =_Q E_Q[X|\mathcal{G}] \} \\
&= E_Q[X|\mathcal{G}] - \pi^*(Q),
\end{aligned}$$

where the last equality follows from $Q \in \mathcal{P}_{\mathcal{G}}$ and

$$\begin{aligned}
\pi^*(Q) &= \sup_{\xi \in L_{\mathcal{F}}} \{ E_Q[\xi + E_Q[X - \xi|\mathcal{G}] \mid \mathcal{G}] - \pi(\xi + E_Q[X - \xi|\mathcal{G}]) \} \\
&= \sup_{\eta \in L_{\mathcal{F}}} \{ E_Q[\eta|\mathcal{G}] - \pi(\eta) \mid \eta = \xi + E_Q[X - \xi|\mathcal{G}] \} \\
&\leq \sup_{\xi \in L_{\mathcal{F}}} \{ E_Q[\xi|\mathcal{G}] - \pi(\xi) \mid E_Q[\xi|\mathcal{G}] =_Q E_Q[X|\mathcal{G}] \} \leq \pi^*(Q).
\end{aligned}$$

■

3.3 On $H(X)$ and a first approximation

For $X \in L_{\mathcal{F}}$ we set

$$H(X) := \sup_{Q \in L_{\mathcal{F}}^* \cap \mathcal{P}} K(X, Q) = \sup_{Q \in L_{\mathcal{F}}^* \cap \mathcal{P}} \inf_{\xi \in L_{\mathcal{F}}} \{ \pi(\xi) \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}] \}$$

and notice that for all $A \in \mathcal{G}$

$$H(X)\mathbf{1}_A = \sup_{Q \in L_{\mathcal{F}}^* \cap \mathcal{P}} \inf_{\xi \in L_{\mathcal{F}}} \{ \pi(\xi)\mathbf{1}_A \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}] \}.$$

Lemma 20 *Properties of $H(X)$. Let $X \in L_{\mathcal{F}}$.*

- i) H is monotone
- ii) $H(X\mathbf{1}_A)\mathbf{1}_A = H(X)\mathbf{1}_A$ for any $A \in \mathcal{G}$.
- iii) There exist a sequence $\{Q^k\}_{k \geq 1} \in L_{\mathcal{F}}^*$ and, for each $k \geq 1$, a sequence $\{\xi_m^{Q^k}\}_{m \geq 1} \in L_{\mathcal{F}}$ satisfying $E_{Q^k}[\xi_m^{Q^k} \mid \mathcal{G}] \geq_{Q^k} E_{Q^k}[X|\mathcal{G}]$ and

$$\pi(\xi_m^{Q^k}) \downarrow K(X, Q^k) \text{ as } m \uparrow \infty, \quad K(X, Q^k) \uparrow H(X) \text{ as } k \uparrow \infty, \quad (12)$$

$$H(X) = \lim_{k \rightarrow \infty} \lim_{m \rightarrow \infty} \pi(\xi_m^{Q^k}). \quad (13)$$

Proof. i) is trivial; ii) follows applying the same argument used in equation (10); the other property is an immediate consequence of what proved in Lemma 19 and 17 regarding the properties of being downward directed and upward directed. ■

Lemma 21 *Let $\mathcal{Q} \subseteq L_F^* \cap \mathcal{P}$ and suppose that the map $S : L_{\mathcal{G}} \times \mathcal{Q} \rightarrow L_{\mathcal{G}}$ is quasiconvex with respect to $Y \in L_{\mathcal{G}}$, for each $Q \in \mathcal{Q}$. Then the functional*

$$f(X) = \sup_{Q \in \mathcal{Q}} S(E_Q[X|\mathcal{G}], Q)$$

is quasiconvex with respect to $X \in L_{\mathcal{F}}$. In particular, $H(X)$ is quasiconvex with respect to $X \in L_{\mathcal{F}}$.

Proof. The first claim is a straightforward application of the definition. By Lemma 19 i) $K(\cdot, Q)$ is quasiconvex and the second statement follows. ■

The following Proposition is an uniform approximation result which stands under stronger assumptions, that are satisfied, for example, by L^p spaces, $p \in [1, +\infty]$. We will not use this Proposition in the proof of Theorem 8, even though it can be useful for understanding the heuristic outline of its proof, as sketched in Section 4.1.

Proposition 22 *Suppose that $L_F^* \hookrightarrow L_{\mathcal{F}}^1$ is a Banach Lattice with the property: for any sequence $\{\eta_n\}_n \subseteq (L_F^*)_+$, $\eta_n \eta_m = 0$ for every $n \neq m$, there exists a sequence $\{\alpha_k\}_k \subset (0, +\infty)$ such that $\sum_n \alpha_n \eta_n \in (L_F^*)_+$. If $H(X) > -\infty$ \mathbb{P} -a.s., then for every $\varepsilon > 0$ there exists $Q_\varepsilon \in L_F^* \cap \mathcal{P}$ such that*

$$H(X) - K(X, Q_\varepsilon) < \varepsilon \quad (14)$$

Proof. From Lemma 20, eq. (12), we know that there exists a sequence $Q_k \in L_F^* \cap \mathcal{P}$ such that:

$$K(X, Q_k) \uparrow H(X), \text{ as } k \uparrow \infty.$$

Define for each $k \geq 1$ the sets

$$D_k =: \{H(X) - K(X, Q_k) \leq \varepsilon\}$$

and note that

$$\mathbb{P}(D_k) \uparrow 1 \text{ as } k \uparrow \infty. \quad (15)$$

Consider the disjoint family $\{F_k\}_{k \geq 1}$ of \mathcal{G} -measurable sets: $F_1 = D_1$, $F_k = D_k \setminus D_{k-1}$, $k \geq 2$. By induction one easily shows that $\bigcup_{k=1}^n F_k = D_n$ for all $n \geq 1$.

This and (15) imply that $\mathbb{P}\left(\bigcup_{k=1}^{\infty} F_k\right) = 1$. Consider the sequence $\left\{\frac{dQ_k}{d\mathbb{P}} \mathbf{1}_{F_k}\right\}$. From the assumption on L_F^* we may find a sequence $\{\alpha_k\}_k \subset (0, +\infty)$ such that

$\frac{d\tilde{Q}_\epsilon}{d\mathbb{P}} =: \sum_{k=1}^{\infty} \alpha_k \frac{dQ_k}{d\mathbb{P}} \mathbf{1}_{F_k} \in L_{\mathcal{F}}^* \hookrightarrow L_{\mathcal{F}}^1$. Hence, $\tilde{Q}_\epsilon \in (L_{\mathcal{F}}^*)_+ \cap (L_{\mathcal{F}}^1)_+$ and, since $\{F_k\}_{k \geq 1}$ are disjoint,

$$\frac{d\tilde{Q}_\epsilon}{d\mathbb{P}} \mathbf{1}_{F_k} = \alpha_k \frac{dQ_k}{d\mathbb{P}} \mathbf{1}_{F_k}, \text{ for any } k \geq 1.$$

Normalize \tilde{Q}_ϵ and denote with $Q_\epsilon = \lambda \tilde{Q}_\epsilon \in L_{\mathcal{F}}^* \cap \mathcal{P}$ the element satisfying $\|\frac{dQ_\epsilon}{d\mathbb{P}}\|_{L_F^1} = 1$. Applying Lemma 19 (vi) we deduce that for any $k \geq 1$

$$K(X, Q_\epsilon) \mathbf{1}_{F_k} = K(X, \tilde{Q}_\epsilon) \mathbf{1}_{F_k} = K(X, \alpha_k Q_k) \mathbf{1}_{F_k} = K(X, Q_k) \mathbf{1}_{F_k},$$

and

$$H(X) \mathbf{1}_{F_k} - K(X, Q_\epsilon) \mathbf{1}_{F_k} = H(X) \mathbf{1}_{F_k} - K(X, Q_k) \mathbf{1}_{F_k} \leq \epsilon \mathbf{1}_{F_k}.$$

The condition (14) is then a consequence of $\mathbb{P}(D_k) \uparrow 1$. Notice that the assumption $H(X) > -\infty$ is only used in (15) and that without this assumption the conclusion (14) would hold true on the set $\{H(X) > -\infty\}$. ■

3.4 On the map π_A

Consider the following

Definition 23 Given $\pi : L_{\mathcal{F}} \rightarrow L_{\mathcal{G}}$ we define for every $A \in \mathcal{G}$ the map

$$\pi_A : L_{\mathcal{F}} \rightarrow \overline{\mathbb{R}} \text{ by } \pi_A(X) := \operatorname{ess\,sup}_{\omega \in A} \pi(X)(\omega).$$

Notice that the map π_A inherits from π the properties (MON), (QCO) and (CFB). Applying Proposition 13 we deduce that π_A is also $\sigma(L_{\mathcal{F}}, L_{\mathcal{F}}^*)$ -lsc.

Proposition 24 Under the same assumptions of Theorem 8 if $A \in \mathcal{G}$

$$\pi_A(X) = \sup_{Q \in L_{\mathcal{F}}^* \cap \mathcal{P}} \inf_{\xi \in L_{\mathcal{F}}} \{\pi_A(\xi) \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}]\}. \quad (16)$$

Proof. From $L_{\mathcal{F}}^* \hookrightarrow L^1(\mathcal{F})$, we have: $L_{\mathcal{F}}^* \cap \mathcal{P} = \left\{ \frac{dQ}{d\mathbb{P}} \mid Q \in (L_{\mathcal{F}})_+^* \text{ and } Q(1) = 1 \right\}$. Since π_A is $\sigma(L_{\mathcal{F}}, L_{\mathcal{F}}^*)$ -lsc the representation (16) follows immediately applying Proposition 33 to the map π_A and observing that

$$\begin{aligned} \pi_A(X) &= \sup_{Q \in L_{\mathcal{F}}^* \cap \mathcal{P}} \inf_{\xi \in L_{\mathcal{F}}} \{\pi_A(\xi) \mid E_Q[\xi] \geq E_Q[X]\} \\ &\leq \sup_{Q \in L_{\mathcal{F}}^* \cap \mathcal{P}} \inf_{\xi \in L_{\mathcal{F}}} \{\pi_A(\xi) \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}]\} \leq \pi_A(X). \end{aligned}$$

■

4 Proof of the main results

Notations: In the following, we will only consider *finite* partitions $\Gamma = \{A^\Gamma\}$ of \mathcal{G} measurable sets $A^\Gamma \in \Gamma$ and we set

$$\begin{aligned}\pi^\Gamma(X) &: = \sum_{A^\Gamma \in \Gamma} \pi_{A^\Gamma}(X) \mathbf{1}_{A^\Gamma}, \\ K^\Gamma(X, Q) &: = \inf_{\xi \in L_{\mathcal{F}}} \{ \pi^\Gamma(\xi) \mid E_Q[\xi | \mathcal{G}] \geq_Q E_Q[X | \mathcal{G}] \} \\ H^\Gamma(X) &: = \sup_{Q \in L_{\mathcal{F}}^* \cap \mathcal{P}} K^\Gamma(X, Q)\end{aligned}$$

4.1 Outline of the proof

We anticipate an heuristic sketch of the proof of Theorem 8, pointing out the essential arguments involved in it and we defer to the following section the details and the rigorous statements.

The proof relies on the equivalence of the following conditions:

1. $\pi(X) = H(X)$.
2. $\forall \varepsilon > 0, \exists Q_\varepsilon \in L_{\mathcal{F}}^* \cap \mathcal{P}$ such that $\pi(X) - K(X, Q_\varepsilon) < \varepsilon$.
3. $\forall \varepsilon > 0, \exists Q_\varepsilon \in L_{\mathcal{F}}^* \cap \mathcal{P}$ such that

$$\{\xi \in L_{\mathcal{F}} \mid E_{Q_\varepsilon}[\xi | \mathcal{G}] \geq_{Q_\varepsilon} E_{Q_\varepsilon}[X | \mathcal{G}]\} \subseteq \{\xi \in L_{\mathcal{F}} \mid \pi(\xi) > \pi(X) - \varepsilon\}. \quad (17)$$

Indeed, 1. \Rightarrow 2. is a consequence of Proposition 22 (when it holds true); 2. \Rightarrow 3. follows from the observation that $\pi(X) < K(X, Q_\varepsilon) + \varepsilon$ implies $\pi(X) < \pi(\xi) + \varepsilon$ for every ξ satisfying $E_{Q_\varepsilon}[\xi | \mathcal{G}] \geq_{Q_\varepsilon} E_{Q_\varepsilon}[X | \mathcal{G}]$; 3. \Rightarrow 1. is implied by the inequalities:

$$\begin{aligned}\pi(X) - \varepsilon &\leq \inf\{\pi(\xi) \mid \pi(\xi) > \pi(X) - \varepsilon\} \\ &\leq \inf_{\xi \in L_{\mathcal{F}}} \{\pi(\xi) \mid E_{Q_\varepsilon}[\xi | \mathcal{G}] \geq_{Q_\varepsilon} E_{Q_\varepsilon}[X | \mathcal{G}]\} \leq H(X) \leq \pi(X).\end{aligned}$$

Unfortunately, we cannot prove Item 3. directly, relying on Hahn-Banach Theorem, as it happened in the real case (see the proof of Theorem 1, equation (53), in Appendix). Indeed, the complement of the set in the RHS of (17) is not any more a convex set - unless π is real valued - regardless of the continuity assumption made on π .

Also the method applied in the conditional convex case [DS05] can not be used here, since the map $X \rightarrow E_{\mathbb{P}}[\pi(X)]$ there adopted preserves convexity but not quasiconvexity.

The idea is then to apply an approximation argument and the choice of approximating $\pi(\cdot)$ by $\pi^\Gamma(\cdot)$, is forced by the need to preserve quasiconvexity.

- I The first step is to prove (see Proposition 28) that: $H^\Gamma(X) = \pi^\Gamma(X)$. This is based on the representation of the *real valued* quasiconvex map π_A in Proposition 24. Therefore, the assumptions (LSC), (MON), (REG) and (QCO) on π are here all needed.
- II Then it is a simple matter to deduce $\pi(X) = \inf_\Gamma \pi^\Gamma(X) = \inf_\Gamma H^\Gamma(X)$, where the inf is taken with respect to all finite partitions.
- III As anticipated in (3), the last step, i.e. proving that $\inf_\Gamma H^\Gamma(X) = H(X)$, is more delicate. It can be shown easily that is possible to approximate $H(X)$ with $K(X, Q_\varepsilon)$ on a set A_ε of probability arbitrarily close to 1. However, we need the following *uniform* approximation: For any $\varepsilon > 0$ there exists $Q_\varepsilon \in L_{\mathcal{F}}^* \cap \mathcal{P}$ such that for any finite partition Γ we have $H^\Gamma(X) - K^\Gamma(X, Q_\varepsilon) < \varepsilon$ on the same set A_ε . This key approximation result, based on Lemma 27, shows that the element Q_ε does not depend on the partition and allows us (see equation (24)) to conclude the proof.

4.2 Details

The following two lemmas are applications of measure theory

Lemma 25 *For every $Y \in L_{\mathcal{G}}^0$ there exists a sequence $\Gamma(n)$ of finite partitions such that $\sum_{\Gamma(n)} (\sup_{A^{\Gamma(n)}} Y) \mathbf{1}_{A^{\Gamma(n)}}$ converges in probability, and \mathbb{P} -a.s., to Y .*

Proof. Fix $\varepsilon, \delta > 0$ and consider the partitions $\Gamma(n) = \{A_0^n, A_1^n, \dots, A_{n2^{n+1}+1}^n\}$ where

$$\begin{aligned} A_0^n &= \{Y \in (-\infty, -n]\} \\ A_j^n &= \{Y \in (-n + \frac{j-1}{2^n}, -n + \frac{j}{2^n}]\} \quad \forall j = 1, \dots, n2^{n+1} \\ A_{n2^{n+1}+1}^n &= \{Y \in (n, +\infty)\} \end{aligned}$$

Since $\mathbb{P}(A_0^n \cup A_{n2^{n+1}+1}^n) \rightarrow 0$ as $n \rightarrow \infty$, we consider N such that $\mathbb{P}(A_0^N \cup A_{N2^{N+1}+1}^N) \leq 1 - \varepsilon$. Moreover we may find M such that $\frac{1}{2^M} < \delta$, and hence for $\Gamma = \Gamma(M \vee N)$ we have:

$$\mathbb{P} \left\{ \omega \in \Omega \mid \sum_{A^\Gamma \in \Gamma} \left(\sup_{A^\Gamma} Y \right) \mathbf{1}_{A^\Gamma}(\omega) - Y(\omega) < \delta \right\} > 1 - \varepsilon. \quad (18)$$

■

Lemma 26 *For each $X \in L_{\mathcal{F}}$ and $Q \in L_{\mathcal{F}}^* \cap \mathcal{P}$*

$$\inf_{\Gamma} K^\Gamma(X, Q) = K(X, Q)$$

where the infimum is taken with respect to all finite partitions Γ .

Proof.

$$\begin{aligned}
\inf_{\Gamma} K^{\Gamma}(X, Q) &= \inf_{\Gamma} \inf_{\xi \in L_{\mathcal{F}}} \{ \pi^{\Gamma}(\xi) \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}] \} \\
&= \inf_{\xi \in L_{\mathcal{F}}} \left\{ \inf_{\Gamma} \pi^{\Gamma}(\xi) \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}] \right\} \\
&= \inf_{\xi \in L_{\mathcal{F}}} \{ \pi(\xi) \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}] \} = K(X, Q). \quad (19)
\end{aligned}$$

where the first equality in (19) follows from the convergence shown in Lemma 25. ■

The following already mentioned key result is proved in the Appendix, for it needs a pretty long argument.

Lemma 27 *Let $X \in L_{\mathcal{F}}$ and let P and Q be arbitrary elements of $L_{\mathcal{F}}^* \cap \mathcal{P}$. Suppose that there exists $B \in \mathcal{G}$ satisfying: $K(X, P)\mathbf{1}_B > -\infty$, $\pi_B(X) < +\infty$ and*

$$K(X, Q)\mathbf{1}_B \leq K(X, P)\mathbf{1}_B + \varepsilon\mathbf{1}_B,$$

for some $\varepsilon \geq 0$. Then for every partition $\Gamma = \{B^C, \tilde{\Gamma}\}$, where $\tilde{\Gamma}$ is a partition of B , we have

$$K^{\Gamma}(X, Q)\mathbf{1}_B \leq K^{\Gamma}(X, P)\mathbf{1}_B + \varepsilon\mathbf{1}_B.$$

Since π^{Γ} assumes only a finite number of values, we may apply Proposition 24 and deduce the dual representation of π^{Γ} .

Proposition 28 *Suppose that the assumptions of Theorem 8 hold true and Γ is a finite partition. If for every $X \in L_{\mathcal{F}}$, $\pi^{\Gamma}(X) < +\infty$ then:*

$$H^{\Gamma}(X) = \pi^{\Gamma}(X) \geq \pi(X) \quad (20)$$

and therefore

$$\inf_{\Gamma} H^{\Gamma}(X) = \pi(X).$$

Proof. First notice that $K^{\Gamma}(X, Q) \leq H^{\Gamma}(X) \leq \pi^{\Gamma}(X) < +\infty$ for all $Q \in L_{\mathcal{F}}^* \cap \mathcal{P}$. Consider the sigma algebra $\mathcal{G}^{\Gamma} := \sigma(\Gamma) \subseteq \mathcal{G}$, generated by the finite partition Γ . Hence from Proposition 24 we have for every $A^{\Gamma} \in \Gamma$

$$\pi_{A^{\Gamma}}(X) = \sup_{Q \in L_{\mathcal{F}}^* \cap \mathcal{P}} \inf_{\xi \in L_{\mathcal{F}}} \{ \pi_{A^{\Gamma}}(\xi) \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}] \}. \quad (21)$$

Moreover $H^{\Gamma}(X)$ is constant on A^{Γ} since it is \mathcal{G}^{Γ} -measurable as well. Using the fact that $\pi^{\Gamma}(\cdot)$ is constant on each A^{Γ} , for every $A^{\Gamma} \in \Gamma$ we then have:

$$\begin{aligned}
H^{\Gamma}(X)\mathbf{1}_{A^{\Gamma}} &= \sup_{Q \in L_{\mathcal{F}}^* \cap \mathcal{P}} \inf_{\xi \in L_{\mathcal{F}}} \{ \pi^{\Gamma}(\xi)\mathbf{1}_{A^{\Gamma}} \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}] \} \\
&= \sup_{Q \in L_{\mathcal{F}}^* \cap \mathcal{P}} \inf_{\xi \in L_{\mathcal{F}}} \{ \pi_{A^{\Gamma}}(\xi)\mathbf{1}_{A^{\Gamma}} \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}] \} \\
&= \pi_{A^{\Gamma}}(X)\mathbf{1}_{A^{\Gamma}} = \pi^{\Gamma}(X)\mathbf{1}_{A^{\Gamma}} \quad (22)
\end{aligned}$$

where the first equality in (22) follows from (21). The remaining statement is a consequence of (20) and Lemma 25 ■

Proof of Theorem 8. Obviously $\pi(X) \geq H(X)$, since X satisfies the constraints in the definition of $H(X)$. We may assume w.l.o.g. that $\pi(0) = 0$ and so $\pi(X\mathbf{1}_G) = \pi(X)\mathbf{1}_G$ for every $G \in \mathcal{G}$ (indeed, otherwise we could consider $\rho(\cdot) = \pi(\cdot) - \pi(0)$).

First we assume that π is uniformly bounded, i.e. there exists $c > 0$ such that for all $X \in L_{\mathcal{F}}$ $|\pi(X)| \leq c$. Then $H(X) > -\infty$.

From Lemma 20, eq. (12), we know that there exists a sequence $Q_k \in L_{\mathcal{F}}^* \cap \mathcal{P}$ such that:

$$K(X, Q_k) \uparrow H(X), \text{ as } k \uparrow \infty.$$

Therefore, for any $\varepsilon > 0$ we may find $Q_\varepsilon \in L_{\mathcal{F}}^* \cap \mathcal{P}$ and $A_\varepsilon \in \mathcal{G}$, $\mathbb{P}(A_\varepsilon) > 1 - \varepsilon$ such that

$$H(X)\mathbf{1}_{A_\varepsilon} - K(X, Q_\varepsilon)\mathbf{1}_{A_\varepsilon} \leq \varepsilon\mathbf{1}_{A_\varepsilon}.$$

Since $H(X) \geq K(X, Q) \forall Q \in L_{\mathcal{F}}^* \cap \mathcal{P}$,

$$(K(X, Q_\varepsilon) + \varepsilon)\mathbf{1}_{A_\varepsilon} \geq K(X, Q)\mathbf{1}_{A_\varepsilon} \quad \forall Q \in L_{\mathcal{F}}^* \cap \mathcal{P}.$$

This is the basic inequality that enable us to apply Lemma 27, replacing there P with Q_ε and B with A_ε . Only notice that $\sup_{\Omega} \pi(X) \leq c$ and $K(X, Q) > -\infty$ for every $Q \in L_{\mathcal{F}}^* \cap \mathcal{P}$. This Lemma assures that for every partition Γ of Ω

$$(K^\Gamma(X, Q_\varepsilon) + \varepsilon)\mathbf{1}_{A_\varepsilon} \geq K^\Gamma(X, Q)\mathbf{1}_{A_\varepsilon} \quad \forall Q \in L_{\mathcal{F}}^* \cap \mathcal{P}. \quad (23)$$

From the definition of *essential supremum* of a class of r.v. equation (23) implies that for every Γ

$$(K^\Gamma(X, Q_\varepsilon) + \varepsilon)\mathbf{1}_{A_\varepsilon} \geq \sup_{Q \in L_{\mathcal{F}}^* \cap \mathcal{P}} K^\Gamma(X, Q)\mathbf{1}_{A_\varepsilon} = H^\Gamma(X)\mathbf{1}_{A_\varepsilon}. \quad (24)$$

Since $\pi^\Gamma \leq c$, applying Proposition 28, equation (20), we get

$$(K^\Gamma(X, Q_\varepsilon) + \varepsilon)\mathbf{1}_{A_\varepsilon} \geq \pi(X)\mathbf{1}_{A_\varepsilon}.$$

Taking the *infimum* over all possible partitions, as in Lemma 26, we deduce:

$$(K(X, Q_\varepsilon) + \varepsilon)\mathbf{1}_{A_\varepsilon} \geq \pi(X)\mathbf{1}_{A_\varepsilon}. \quad (25)$$

Hence, for any $\varepsilon > 0$

$$(K(X, Q_\varepsilon) + \varepsilon)\mathbf{1}_{A_\varepsilon} \geq \pi(X)\mathbf{1}_{A_\varepsilon} \geq H(X)\mathbf{1}_{A_\varepsilon} \geq K(X, Q_\varepsilon)\mathbf{1}_{A_\varepsilon}$$

which implies $\pi(X) = H(X)$, since $\mathbb{P}(A_\varepsilon) \rightarrow 1$ as $\varepsilon \rightarrow 0$.

Now we consider the case when π is not necessarily bounded. We define the new map $\psi(\cdot) := \arctan(\pi(\cdot))$ and notice that $\psi(\xi)$ is a \mathcal{G} -measurable r.v. satisfying $|\psi(X)| \leq \frac{\pi}{2}$ for every $X \in L_{\mathcal{F}}$. Moreover ψ is (MON), (QCO), (LSC)

and $\psi(X\mathbf{1}_G) = \psi(X)\mathbf{1}_G$ for every $G \in \mathcal{G}$. Since ψ is bounded, by the above argument we may conclude that

$$\psi(X) = H_\psi(X) := \sup_{Q \in L_{\mathcal{F}}^* \cap \mathcal{P}} K_\psi(X, Q)$$

where

$$K_\psi(X, Q) := \inf_{\xi \in L_{\mathcal{F}}} \{ \psi(\xi) \mid E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}] \}.$$

Applying again Lemma 20, equation (12), there exists $Q^k \in L_{\mathcal{F}}^*$ such that

$$H_\psi(X) = \lim_k K_\psi(X, Q^k).$$

We will show below that

$$K_\psi(X, Q^k) = \arctan K(X, Q^k). \quad (26)$$

Admitting this, we have for \mathbb{P} -almost every $\omega \in \Omega$

$$\begin{aligned} \arctan(\pi(X)(\omega)) &= \psi(X)(\omega) = H_\psi(X)(\omega) = \lim_k K_\psi(X, Q^k)(\omega) \\ &= \lim_k \arctan K(X, Q^k)(\omega) = \arctan(\lim_k K(X, Q^k)(\omega)), \end{aligned}$$

where we used the continuity of the function \arctan . This implies $\pi(X) = \lim_k K(X, Q^k)$ and we conclude:

$$\pi(X) = \lim_k K(X, Q^k) \leq H(X) \leq \pi(X).$$

It only remains to show (26). We prove that for every fixed $Q \in L_{\mathcal{F}}^* \cap \mathcal{P}$

$$K_\psi(X, Q) = \arctan(K(X, Q)).$$

Since π and ψ are regular, from Lemma 19 iv), there exist $\xi_h^Q \in L_{\mathcal{F}}$ and $\eta_h^Q \in L_{\mathcal{F}}$ such that

$$E_Q[\xi_h^Q|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}], \quad E_Q[\eta_h^Q|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}], \quad \forall h \geq 1, \quad (27)$$

$\psi(\xi_h^Q) \downarrow K_\psi(X, Q)$ and $\pi(\eta_h^Q) \downarrow K(X, Q)$, as $h \uparrow \infty$. From (27) and the definitions of $K(X, Q)$, $K_\psi(X, Q)$ and by the continuity and monotonicity of \arctan we get:

$$\begin{aligned} K_\psi(X, Q) &\leq \lim_h \psi(\eta_h^Q) = \lim_h \arctan \pi(\eta_h^Q) = \arctan \lim_h \pi(\eta_h^Q) \\ &= \arctan K(X, Q) \leq \arctan \lim_h \pi(\xi_h^Q) = \lim_h \psi(\xi_h^Q) = K_\psi(X, Q). \end{aligned}$$

■

Remark 29 Consider $Q \in \mathcal{P}$ such that $Q \sim \mathbb{P}$ on \mathcal{G} and define the new probability

$$\tilde{Q}(F) := E_Q \left[\frac{d\mathbb{P}^{\mathcal{G}}}{dQ} \mathbf{1}_F \right] \quad \text{where} \quad \frac{d\mathbb{P}^{\mathcal{G}}}{dQ} =: E_Q \left[\frac{d\mathbb{P}}{dQ} \middle| \mathcal{G} \right], \quad F \in \mathcal{F}.$$

Then $\tilde{Q}(G) = \mathbb{P}(G)$ for all $G \in \mathcal{G}$, and so $\tilde{Q} \in \mathcal{P}_{\mathcal{G}}$. Moreover, it is easy to check that for all $X \in L_{\mathcal{F}}$ and $Q \in L_{\mathcal{F}}^* \cap \mathcal{P}$ such that $Q \sim \mathbb{P}$ on \mathcal{G} we have:

$$E_{\tilde{Q}}[X|\mathcal{G}] = E_Q[X|\mathcal{G}]$$

which implies $K(X, \tilde{Q}) = K(X, Q)$.

Proof of Corollary 9. Consider the probability $Q_{\varepsilon} \in L_{\mathcal{F}}^* \cap \mathcal{P}$ built up in Theorem 8, equation (25). We claim that Q_{ε} is equivalent to \mathbb{P} on A_{ε} . By contradiction there exists $B \in \mathcal{G}$, $B \subseteq A_{\varepsilon}$, such that $\mathbb{P}(B) > 0$ but $Q_{\varepsilon}(B) = 0$. Consider $\eta \in L_{\mathcal{F}}$, $\delta > 0$ such that $\mathbb{P}(\pi(\eta) + \delta < \pi(X)) = 1$ and define $\xi = X\mathbf{1}_{B^c} + \eta\mathbf{1}_B$ so that $E_{Q_{\varepsilon}}[\xi|\mathcal{G}] \geq_{Q_{\varepsilon}} E_{Q_{\varepsilon}}[X|\mathcal{G}]$. By regularity $\pi(\xi) = \pi(X)\mathbf{1}_{B^c} + \pi(\eta)\mathbf{1}_B$ which implies for \mathbb{P} -a.e. $\omega \in B$

$$\pi(\xi)(\omega) + \delta = \pi(\eta)(\omega) + \delta < \pi(X)(\omega) \leq K(X, Q_{\varepsilon})(\omega) + \varepsilon \leq \pi(\xi)(\omega) + \varepsilon$$

which is impossible for $\varepsilon \leq \delta$. So $Q_{\varepsilon} \sim \mathbb{P}$ on A_{ε} for all small $\varepsilon \leq \delta$.

Consider \tilde{Q}_{ε} such that $\frac{d\tilde{Q}_{\varepsilon}}{d\mathbb{P}} = \frac{dQ_{\varepsilon}}{d\mathbb{P}}\mathbf{1}_{A_{\varepsilon}} + \frac{d\mathbb{P}}{d\mathbb{P}}\mathbf{1}_{(A_{\varepsilon})^c}$. Up to a normalization factor $\tilde{Q}_{\varepsilon} \in L_{\mathcal{F}}^* \cap \mathcal{P}$ and is equivalent to \mathbb{P} . Moreover from Lemma 19 (vi), $K(X, \tilde{Q}_{\varepsilon})\mathbf{1}_{A_{\varepsilon}} = K(X, Q_{\varepsilon})\mathbf{1}_{A_{\varepsilon}}$ and from Remark 29 we may define $\tilde{Q}_{\varepsilon} \in \mathcal{P}_{\mathcal{G}}$ such that $K(X, \tilde{Q}_{\varepsilon})\mathbf{1}_{A_{\varepsilon}} = K(X, \tilde{Q}_{\varepsilon})\mathbf{1}_{A_{\varepsilon}} = K(X, Q_{\varepsilon})\mathbf{1}_{A_{\varepsilon}}$. From (25) we finally deduce: $K(X, \tilde{Q}_{\varepsilon})\mathbf{1}_{A_{\varepsilon}} + \varepsilon\mathbf{1}_{A_{\varepsilon}} \geq \pi(X)\mathbf{1}_{A_{\varepsilon}}$, and the thesis then follows from $\tilde{Q}_{\varepsilon} \in \mathcal{P}_{\mathcal{G}}$. ■

5 Appendix

5.1 Proof of the key approximation Lemma 27

We will adopt the following notations: If Γ_1 and Γ_2 are two finite partitions of \mathcal{G} -measurable sets then $\Gamma_1 \cap \Gamma_2 := \{A_1 \cap A_2 \mid A_i \in \Gamma_i, i = 1, 2\}$ is a finite partition finer than each Γ_1 and Γ_2 .

Lemma 30 is the natural generalization of Lemma 17 to the approximated problem.

Lemma 30 For every partition Γ , $X \in L_{\mathcal{F}}$ and $Q \in L_{\mathcal{F}}^* \cap \mathcal{P}$, the set

$$\mathcal{A}_Q^{\Gamma}(X) \doteq \{\pi^{\Gamma}(\xi) \mid \xi \in L_{\mathcal{F}} \text{ and } E_Q[\xi|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}]\}$$

is downward directed. This implies that there exists a sequence $\{\eta_m^Q\}_{m=1}^{\infty} \in L_{\mathcal{F}}$ such that

$$E_Q[\eta_m^Q|\mathcal{G}] \geq_Q E_Q[X|\mathcal{G}] \quad \forall m \geq 1, \quad \pi^{\Gamma}(\eta_m^Q) \downarrow K^{\Gamma}(X, Q) \text{ as } m \uparrow \infty.$$

Proof. To show that the set $\mathcal{A}_Q^\Gamma(X)$ is downward directed we use the notations and the results in the proof of Lemma 17 and check that

$$\pi^\Gamma(\xi^*) = \pi^\Gamma(\xi_1 \mathbf{1}_G + \xi_2 \mathbf{1}_{G^c}) \leq \min \{ \pi^\Gamma(\xi_1), \pi^\Gamma(\xi_2) \}.$$

■

Now we show that for any given sequence of partition there exists one sequence that works for all.

Lemma 31 *For any fixed, at most countable, family of partitions $\{\Gamma(h)\}_{h \geq 1}$ and $Q \in L_{\mathcal{F}}^* \cap \mathcal{P}$, there exists a sequence $\{\xi_m^Q\}_{m=1}^\infty \in L_{\mathcal{F}}$ such that*

$$\begin{aligned} E_Q[\xi_m^Q | \mathcal{G}] &\geq_Q E_Q[X | \mathcal{G}] \quad \text{for all } m \geq 1 \\ \pi(\xi_m^Q) &\downarrow K(X, Q) \quad \text{as } m \uparrow \infty \\ \text{and for all } h \quad \pi^{\Gamma(h)}(\xi_m^Q) &\downarrow K^{\Gamma(h)}(X, Q) \quad \text{as } m \uparrow \infty. \end{aligned}$$

Proof. Apply Lemma 17 and Lemma 30 and find $\{\varphi_m^0\}_m, \{\varphi_m^1\}_m, \dots, \{\varphi_m^h\}_m, \dots$ such that for every i and m we have $E_Q[\varphi_m^i | \mathcal{G}] \geq_Q E_Q[X | \mathcal{G}]$ and

$$\begin{aligned} \pi(\varphi_m^0) &\downarrow K(X, Q) \quad \text{as } m \uparrow \infty \\ \text{and for all } h \quad \pi^{\Gamma(h)}(\varphi_m^h) &\downarrow K^{\Gamma(h)}(X, Q) \quad \text{as } m \uparrow \infty. \end{aligned}$$

For each $m \geq 1$ consider $\bigwedge_{i=0}^m \pi(\varphi_m^i)$: then there will exists a (non unique) finite partition of Ω , $\{F_m^i\}_{i=1}^m$ such that

$$\bigwedge_{i=0}^m \pi(\varphi_m^i) = \sum_{i=0}^m \pi(\varphi_m^i) \mathbf{1}_{F_m^i}.$$

Denote $\xi_m^Q =: \sum_{i=0}^m \varphi_m^i \mathbf{1}_{F_m^i}$ and notice that $\sum_{i=0}^m \pi(\varphi_m^i) \mathbf{1}_{F_m^i} \stackrel{(REG)}{=} \pi(\xi_m^Q)$ and $E_Q[\xi_m^Q | \mathcal{G}] \geq_Q E_Q[X | \mathcal{G}]$ for every m . Moreover $\pi(\xi_m^Q)$ is decreasing and $\pi(\xi_m^Q) \leq \pi(\varphi_m^0)$ implies $\pi(\xi_m^Q) \downarrow K(X, Q)$. For every fixed h we have $\pi(\xi_m^Q) \leq \pi(\varphi_m^h)$ for all $h \leq m$ and hence:

$$\pi^{\Gamma(h)}(\xi_m^Q) \leq \pi^{\Gamma(h)}(\varphi_m^h) \text{ implies } \pi^{\Gamma(h)}(\xi_m^Q) \downarrow K^{\Gamma(h)}(X, Q) \text{ as } m \uparrow \infty.$$

■

Finally, we state the basic step used in the proof of Lemma 27.

Lemma 32 *Let $X \in L_{\mathcal{F}}$ and let P and Q be arbitrary elements of $L_{\mathcal{F}}^* \cap \mathcal{P}$. Suppose that there exists $B \in \mathcal{G}$ satisfying: $K(X, P) \mathbf{1}_B > -\infty$, $\pi_B(X) < +\infty$ and*

$$K(X, Q) \mathbf{1}_B \leq K(X, P) \mathbf{1}_B + \varepsilon \mathbf{1}_B,$$

for some $\varepsilon \geq 0$. Then for any $\delta > 0$ and any partition Γ_0 there exists $\Gamma \supseteq \Gamma_0$ for which

$$K^\Gamma(X, Q) \mathbf{1}_B \leq K^\Gamma(X, P) \mathbf{1}_B + \varepsilon \mathbf{1}_B + \delta \mathbf{1}_B$$

Proof. By our assumptions we have: $-\infty < K(X, P)\mathbf{1}_B \leq \pi_B(X) < +\infty$ and $K(X, Q)\mathbf{1}_B \leq \pi_B(X) < +\infty$. Fix $\delta > 0$ and the partition Γ_0 . Suppose by contradiction that for any $\Gamma \supseteq \Gamma_0$ we have $\mathbb{P}(C) > 0$ where

$$C = \{\omega \in B \mid K^\Gamma(X, Q)(\omega) > K^\Gamma(X, P)(\omega) + \varepsilon + \delta\}. \quad (28)$$

Notice that C is the union of a finite number of elements in the partition Γ .

Consider that Lemma 19 guarantees the existence of $\{\xi_h^Q\}_{h=1}^\infty \in L_{\mathcal{F}}$ satisfying:

$$\pi(\xi_h^Q) \downarrow K(X, Q), \text{ as } h \uparrow \infty, \quad E_Q[\xi_h^Q | \mathcal{G}] \geq_Q E_Q[X | \mathcal{G}] \quad \forall h \geq 1. \quad (29)$$

Moreover, for each partition Γ and $h \geq 1$ define:

$$D_h^\Gamma := \left\{ \omega \in \Omega \mid \pi^\Gamma(\xi_h^Q)(\omega) - \pi(\xi_h^Q)(\omega) < \frac{\delta}{4} \right\} \in \mathcal{G},$$

and observe that $\pi^\Gamma(\xi_h^Q)$ decreases if we pass to finer partitions. From Lemma 25 equation (18), we deduce that for each $h \geq 1$ there exists a partition $\tilde{\Gamma}(h)$ such that $\mathbb{P}\left(D_h^{\tilde{\Gamma}(h)}\right) \geq 1 - \frac{1}{2^h}$. For every $h \geq 1$ define the new partition $\Gamma(h) = \left(\bigcap_{j=1}^h \tilde{\Gamma}(j)\right) \cap \Gamma_0$ so that for all $h \geq 1$ we have: $\Gamma(h+1) \supseteq \Gamma(h) \supseteq \Gamma_0$, $\mathbb{P}\left(D_h^{\Gamma(h)}\right) \geq 1 - \frac{1}{2^h}$ and

$$\left(\pi(\xi_h^Q) + \frac{\delta}{4}\right) \mathbf{1}_{D_h^{\Gamma(h)}} \geq \left(\pi^{\Gamma(h)}(\xi_h^Q)\right) \mathbf{1}_{D_h^{\Gamma(h)}}, \quad \forall h \geq 1. \quad (30)$$

Lemma 31 guarantees that for the fixed sequence of partitions $\{\Gamma(h)\}_{h \geq 1}$, there exists a sequence $\{\xi_m^P\}_{m=1}^\infty \in L_{\mathcal{F}}$, which does not depend on h , satisfying

$$E_P[\xi_m^P | \mathcal{G}] \geq_P E_P[X | \mathcal{G}] \quad \forall m \geq 1, \quad (31)$$

$$\pi^{\Gamma(h)}(\xi_m^P) \downarrow K^{\Gamma(h)}(X, P), \text{ as } m \uparrow \infty, \quad \forall h \geq 1. \quad (32)$$

For each $m \geq 1$ and $\Gamma(h)$ define:

$$C_m^{\Gamma(h)} := \left\{ \omega \in C \mid \pi^{\Gamma(h)}(\xi_m^P)(\omega) - K^{\Gamma(h)}(X, P)(\omega) \leq \frac{\delta}{4} \right\} \in \mathcal{G}.$$

Since the expressions in the definition of $C_m^{\Gamma(h)}$ assume only a finite number of values, from (32) and from our assumptions, which imply that $K^{\Gamma(h)}(X, P) \geq K(X, P) > -\infty$ on B , we deduce that for each $\Gamma(h)$ there exists an index $m(\Gamma(h))$ such that: $\mathbb{P}\left(C \setminus C_{m(\Gamma(h))}^{\Gamma(h)}\right) = 0$ and

$$K^{\Gamma(h)}(X, P) \mathbf{1}_{C_{m(\Gamma(h))}^{\Gamma(h)}} \geq \left(\pi^{\Gamma(h)}(\xi_{m(\Gamma(h))}^P) - \frac{\delta}{4}\right) \mathbf{1}_{C_{m(\Gamma(h))}^{\Gamma(h)}}, \quad \forall h \geq 1. \quad (33)$$

Set $E_h = D_h^{\Gamma(h)} \cap C_{m(\Gamma(h))}^{\Gamma(h)} \in \mathcal{G}$ and observe that

$$\mathbf{1}_{E_h} \rightarrow \mathbf{1}_C \quad \mathbb{P} - \text{a.s.} \quad (34)$$

From (30) and (33) we then deduce:

$$\left(\pi(\xi_h^Q) + \frac{\delta}{4} \right) \mathbf{1}_{E_h} \geq \left(\pi^{\Gamma(h)}(\xi_h^Q) \right) \mathbf{1}_{E_h}, \quad \forall h \geq 1, \quad (35)$$

$$K^{\Gamma(h)}(X, P) \mathbf{1}_{E_h} \geq \left(\pi^{\Gamma(h)}(\xi_{m(\Gamma(h))}^P) - \frac{\delta}{4} \right) \mathbf{1}_{E_h}, \quad \forall h \geq 1. \quad (36)$$

We then have for any $h \geq 1$

$$\pi(\xi_h^Q) \mathbf{1}_{E_h} + \frac{\delta}{4} \mathbf{1}_{E_h} \geq \left(\pi^{\Gamma(h)}(\xi_h^Q) \right) \mathbf{1}_{E_h} \quad (37)$$

$$\geq K^{\Gamma(h)}(X, Q) \mathbf{1}_{E_h} \quad (38)$$

$$\geq \left(K^{\Gamma(h)}(X, P) + \varepsilon + \delta \right) \mathbf{1}_{E_h} \quad (39)$$

$$\geq \left(\pi^{\Gamma(h)}(\xi_{m(\Gamma(h))}^P) - \frac{\delta}{4} + \varepsilon + \delta \right) \mathbf{1}_{E_h} \quad (40)$$

$$\geq \left(\pi(\xi_{m(\Gamma(h))}^P) + \varepsilon + \frac{3}{4}\delta \right) \mathbf{1}_{E_h}. \quad (41)$$

(in the above chain of inequalities, (37) follows from (35); (38) follows from (29) and the definition of $K^{\Gamma(h)}(X, Q)$; (39) follows from (28); (40) follows from (36); (41) follows from the definition of the maps $\pi_{A^{\Gamma(h)}}$).

Recalling (31) we then get, for each $h \geq 1$,

$$\pi(\xi_h^Q) \mathbf{1}_{E_h} \geq \left(\pi(\xi_{m(\Gamma(h))}^P) + \varepsilon + \frac{\delta}{2} \right) \mathbf{1}_{E_h} \geq \left(K(X, P) + \varepsilon + \frac{\delta}{2} \right) \mathbf{1}_{E_h} > -\infty. \quad (42)$$

From equation (29) and (34) we have $\pi(\xi_h^Q) \mathbf{1}_{E_h} \rightarrow K(X, Q) \mathbf{1}_C$ \mathbb{P} -a.s. as $h \uparrow \infty$ and so from (42)

$$\begin{aligned} \mathbf{1}_C K(X, Q) &= \lim_h \pi(\xi_h^Q) \mathbf{1}_{E_h} \geq \lim_h \mathbf{1}_{E_h} \left(K(X, P) + \varepsilon + \frac{\delta}{2} \right) \\ &= \mathbf{1}_C \left(K(X, P) + \varepsilon + \frac{\delta}{2} \right) \end{aligned}$$

which contradicts the assumption of the Lemma, since $C \subseteq B$ and $\mathbb{P}(C) > 0$. ■

Proof of Lemma 27. First notice that the assumptions of this Lemma are those of Lemma 32. Assume by contradiction that there exists $\Gamma_0 = \{B^C, \tilde{\Gamma}_0\}$, where $\tilde{\Gamma}_0$ is a partition of B , such that

$$\mathbb{P}(\omega \in B \mid K^{\Gamma_0}(X, Q)(\omega) > K^{\Gamma_0}(X, P)(\omega) + \varepsilon) > 0. \quad (43)$$

By our assumptions we have $K^{\Gamma_0}(X, P)\mathbf{1}_B \geq K(X, P)\mathbf{1}_B > -\infty$ and $K^{\Gamma_0}(X, Q)\mathbf{1}_B \leq \pi_B(X)\mathbf{1}_B < +\infty$. Since K^{Γ_0} is constant on every element $A^{\Gamma_0} \in \Gamma_0$, we denote with $K^{A^{\Gamma_0}}(X, Q)$ the value that the random variable $K^{\Gamma_0}(X, Q)$ assumes on A^{Γ_0} . From (43) we deduce that there exists $\hat{A}^{\Gamma_0} \subseteq B$, $\hat{A}^{\Gamma_0} \in \Gamma_0$, such that

$$+\infty > K^{\hat{A}^{\Gamma_0}}(X, Q) > K^{\hat{A}^{\Gamma_0}}(X, P) + \varepsilon > -\infty.$$

Let then $d > 0$ be defined by

$$d =: K^{\hat{A}^{\Gamma_0}}(X, Q) - K^{\hat{A}^{\Gamma_0}}(X, P) - \varepsilon. \quad (44)$$

Apply Lemma 32 with $\delta = \frac{d}{3}$: then there exists $\Gamma \supseteq \Gamma_0$ (w.l.o.g. $\Gamma = \{B^C, \tilde{\Gamma}\}$ where $\tilde{\Gamma} \supseteq \tilde{\Gamma}_0$) such that

$$K^\Gamma(X, Q)\mathbf{1}_B \leq (K^\Gamma(X, P) + \varepsilon + \delta)\mathbf{1}_B. \quad (45)$$

Considering only the two partitions Γ and Γ_0 , we may apply Lemma 31 and conclude that there exist two sequences $\{\xi_h^P\}_{h=1}^\infty \in L_{\mathcal{F}}$ and $\{\xi_h^Q\}_{h=1}^\infty \in L_{\mathcal{F}}$ satisfying as $h \uparrow \infty$:

$$E_P[\xi_h^P | \mathcal{G}] \geq_P E_P[X | \mathcal{G}], \quad \pi^{\Gamma_0}(\xi_h^P) \downarrow K^{\Gamma_0}(X, P), \quad \pi^\Gamma(\xi_h^P) \downarrow K^\Gamma(X, P) \quad (46)$$

$$E_Q[\xi_h^Q | \mathcal{G}] \geq_Q E_Q[X | \mathcal{G}], \quad \pi^{\Gamma_0}(\xi_h^Q) \downarrow K^{\Gamma_0}(X, Q), \quad \pi^\Gamma(\xi_h^Q) \downarrow K^\Gamma(X, Q) \quad (47)$$

Since $K^{\Gamma_0}(X, P)$ is constant and finite on \hat{A}^{Γ_0} , from (46) we may find $h_1 \geq 1$ such that

$$\pi_{\hat{A}^{\Gamma_0}}(\xi_h^P) - K^{\hat{A}^{\Gamma_0}}(X, P) < \frac{d}{2}, \quad \forall h \geq h_1. \quad (48)$$

From equation (44) and (48) we deduce that

$$\pi_{\hat{A}^{\Gamma_0}}(\xi_h^P) < K^{\hat{A}^{\Gamma_0}}(X, P) + \frac{d}{2} = K^{\hat{A}^{\Gamma_0}}(X, Q) - \varepsilon - d + \frac{d}{2}, \quad \forall h \geq h_1,$$

and therefore, knowing from (47) that $K^{\hat{A}^{\Gamma_0}}(X, Q) \leq \pi_{\hat{A}^{\Gamma_0}}(\xi_h^Q)$,

$$\pi_{\hat{A}^{\Gamma_0}}(\xi_h^P) + \frac{d}{2} < \pi_{\hat{A}^{\Gamma_0}}(\xi_h^Q) - \varepsilon \quad \forall h \geq h_1. \quad (49)$$

We now take into account all the sets $A^\Gamma \subseteq \hat{A}^{\Gamma_0} \subseteq B$. For the convergence of $\pi_{A^\Gamma}(\xi_h^Q)$ we distinguish two cases. On those sets A^Γ for which $K^{A^\Gamma}(X, Q) > -\infty$ we may find, from (47), $\bar{h} \geq 1$ such that

$$\pi_{A^\Gamma}(\xi_h^Q) - K^{A^\Gamma}(X, Q) < \frac{\delta}{2} \quad \forall h \geq \bar{h}.$$

Then using (45) and (46) we have

$$\pi_{A^\Gamma}(\xi_h^Q) < K^{A^\Gamma}(X, Q) + \frac{\delta}{2} \leq K^{A^\Gamma}(X, P) + \varepsilon + \delta + \frac{\delta}{2} \leq \pi_{A^\Gamma}(\xi_h^P) + \varepsilon + \delta + \frac{\delta}{2}$$

so that

$$\pi_{A^\Gamma}(\xi_h^Q) < \pi_{A^\Gamma}(\xi_h^P) + \varepsilon + \frac{3\delta}{2} \quad \forall h \geq \bar{h}.$$

On the other hand, on those sets A^Γ for which $K^{A^\Gamma}(X, Q) = -\infty$ the convergence (47) guarantees the existence of $\hat{h} \geq 1$ for which we obtain again:

$$\pi_{A^\Gamma}(\xi_h^Q) < \pi_{A^\Gamma}(\xi_h^P) + \varepsilon + \frac{3\delta}{2} \quad \forall h \geq \hat{h} \quad (50)$$

(notice that $K^\Gamma(X, P) \geq K(X, P)\mathbf{1}_B > -\infty$ and (46) imply that $\pi_{A^\Gamma}(\xi_h^P)$ converges to a finite value, for $A^\Gamma \subseteq B$).

Since the partition Γ is finite there exists $h_2 \geq 1$ such that equation (50) stands for every $A^\Gamma \subseteq \hat{A}^{\Gamma_0}$ and for every $h \geq h_2$ and for our choice of $\delta = \frac{d}{3}$ (50) becomes

$$\pi_{A^\Gamma}(\xi_h^Q) < \pi_{A^\Gamma}(\xi_h^P) + \varepsilon + \frac{d}{2} \quad \forall h \geq h_2 \quad \forall A^\Gamma \subseteq \hat{A}^{\Gamma_0}. \quad (51)$$

Fix $h^* > \max\{h_1, h_2\}$ and consider the value $\pi_{\hat{A}^{\Gamma_0}}(\xi_{h^*}^Q)$. Then among all $A^\Gamma \subseteq \hat{A}^{\Gamma_0}$ we may find $B^\Gamma \subseteq \hat{A}^{\Gamma_0}$ such that $\pi_{B^\Gamma}(\xi_{h^*}^Q) = \pi_{\hat{A}^{\Gamma_0}}(\xi_{h^*}^Q)$. Thus:

$$\pi_{\hat{A}^{\Gamma_0}}(\xi_{h^*}^Q) = \pi_{B^\Gamma}(\xi_{h^*}^Q) \stackrel{(51)}{<} \pi_{B^\Gamma}(\xi_{h^*}^P) + \varepsilon + \frac{d}{2} \leq \pi_{\hat{A}^{\Gamma_0}}(\xi_{h^*}^P) + \varepsilon + \frac{d}{2} \stackrel{(49)}{<} \pi_{\hat{A}^{\Gamma_0}}(\xi_{h^*}^Q).$$

which is a contradiction. ■

5.2 On quasiconvex real valued maps

Proof of Theorem 1. By definition, for any $X' \in L'$, $R(X'(X), X') \leq f(X)$ and therefore

$$\sup_{X' \in L'} R(X'(X), X') \leq f(X), \quad X \in L.$$

Fix any $X \in L$ and take $\varepsilon \in \mathbb{R}$ such that $\varepsilon > 0$. Then X does not belong to the closed convex set $\{\xi \in L : f(\xi) \leq f(X) - \varepsilon\} := \mathcal{C}_\varepsilon$ (if $f(X) = +\infty$, replace the set \mathcal{C}_ε with $\{\xi \in L : f(\xi) \leq M\}$, for any M). By the Hahn Banach theorem there exists a continuous linear functional that strongly separates X and \mathcal{C}_ε , i.e. there exists $\alpha \in \mathbb{R}$ and $X' \in L'$ such that

$$X'_\varepsilon(X) > \alpha > X'_\varepsilon(\xi) \text{ for all } \xi \in \mathcal{C}_\varepsilon. \quad (52)$$

Hence:

$$\{\xi \in L : X'_\varepsilon(\xi) \geq X'_\varepsilon(X)\} \subseteq (\mathcal{C}_\varepsilon)^C = \{\xi \in L : f(\xi) > f(X) - \varepsilon\} \quad (53)$$

and

$$\begin{aligned} f(X) &\geq \sup_{X' \in L'} R(X'(X), X') \geq R(X'_\varepsilon(X), X'_\varepsilon) \\ &= \inf \{f(\xi) \mid \xi \in L \text{ such that } X'_\varepsilon(\xi) \geq X'_\varepsilon(X)\} \\ &\geq \inf \{f(\xi) \mid \xi \in L \text{ satisfying } f(\xi) > f(X) - \varepsilon\} \geq f(X) - \varepsilon. \end{aligned}$$

■

Proposition 33 *Suppose L is a lattice, $L^* = (L, \geq)^*$ is the order continuous dual space satisfying $L^* \hookrightarrow L^1$ and $(L, \sigma(L, L^*))$ is a locally convex TVS. If $f : L \rightarrow \overline{\mathbb{R}}$ is quasiconvex, $\sigma(L, L^*)$ -lsc and monotone increasing then*

$$f(X) = \sup_{Q \in L_+^* | Q(1)=1} R(Q(X), Q).$$

Proof. We apply Theorem 1 to the locally convex TVS $(L, \sigma(L, L^*))$ and deduce:

$$f(X) = \sup_{Z \in L^* \subseteq L^1} R(Z(X), Z).$$

We now adopt the same notations of the proof of Theorem 1 and let $Z \in L$, $Z \geq 0$. Obviously if $\xi \in \mathcal{C}_\varepsilon$ then $\xi - nZ \in \mathcal{C}_\varepsilon$ for every $n \in \mathbb{N}$ and from (52) we deduce:

$$X'_\varepsilon(\xi - nZ) < \alpha < X'_\varepsilon(X) \quad \Rightarrow \quad X'_\varepsilon(Z) > \frac{X'_\varepsilon(\xi - X)}{n}, \quad \forall n \in \mathbb{N}$$

i.e. $X'_\varepsilon \in L_+^* \subseteq L^1$ and $X'_\varepsilon \neq 0$. Hence $X'_\varepsilon(1) = E_{\mathbb{P}}[X'_\varepsilon] > 0$ and we may normalize X' to $X'/X'_\varepsilon(1)$. ■

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